Philipps Universität Marburg

Quadruple covers and Gorenstein stable surfaces with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$

Dissertation

zur Erlangung des Doktorgrades

der Naturwissenschaften (Dr. rer. nat.)

2021

Dem Fachbereich Mathematik und Informatik der Philipps-Universität Marburg
(Hochschulkennziffer 1180) am 06.05.2021 vorgelegt von

Anh Thi Do

geboren in Hungyen, Vietnam

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Zulassung zum Prüfungsverfahren	19.05.2021
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To my beloved family

Abstract

In this thesis we study Gorenstein stable surfaces with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$. These arise as quadruple covers of the projective plane and we give the precise relation between the structure of the cover and the canonical ring. We then use these results to study some strata of the moduli space $\overline{\mathfrak{M}}_{1,2}^{\text{Gor}}$.

Zusammenfassung

In dieser Arbeit studieren wir stabile Gorenstein-Flächen mit Invarianten $K_X^2 = 1$ und $\chi(\mathcal{O}_X) = 2$. Diese entstehen alle als vierfache Überlagerungen der projektiven Ebene und wir beschreiben den genauen Zusammenhang zwischen dem kanonischen Ring und der Struktur der vierfachen Überlagerung. Hiermit gelingt es uns einige Strate im Modulraum $\overline{\mathfrak{M}}_{1,2}^{\text{Gor}}$ zu beschreiben.

Contents

Introduction

It is a general fact that moduli spaces of *nice* objects in algebraic geometry, say smooth varieties, are often non-compact. However, there is usually a modular compactification where the boundary points correspond to related but more complicated objects.

Such a modular compactification has been known for the moduli space $\mathcal{M}_g$ of smooth curves of genus g for a long time and in [Mor] Kollár and Shepherd-Barron made the first step towards the construction of a modular compactification $\overline{\mathfrak{M}}$ for Gieseker's moduli space of (canonical models of) surfaces of general type [Gie77]. Even though the actual construction of the moduli space was delayed for several decades because of formidable technical obstacles to be overcome, it was clear from the beginning that the objects parametrised by $\overline{\mathfrak{M}}$ should be surfaces with semi-log-canonical singularities and ample canonical divisor, for short *stable surfaces*. Nowadays, the existence of the compactification is known, and it is worthwhile to study individual components to get a feeling for the geometry of stable surfaces.

In series of papers, Franciosi, Pardini, and Rollenske [FPR15b, FPR15a, FPR17] realised that under the additional assumption that the canonical divisor is Cartier, that is, the case of Gorenstein stable surfaces, the study of the canonical ring can yield a detailed description, especially for small invariants.

The dissertation's main focus is Gorenstein stable surfaces with $K^2 = 1$ and $\chi = 2$. It had been classically known that canonical models of smooth surfaces with these invariants are quadruple covers of the projective plane [Cat80] and this description extends to Gorenstein stable case [FPR17].

Our study of these surfaces is thus guided by three viewpoints: direct geometric arguments as employed in [FPR17], the structure of the canonical ring, and the structure of the quadruple cover.

For the latter, we build on the theory for Gorenstein covers of degree ≤ 5 laid out by Casnati and Ekedahl in [CE96], which shows that Gorenstein quadruple covers of the plane are embedded as subvarieties of codimension 2 in a $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$, locally given by the intersection of two relative quadrics.

Altogether, we work out the relations and interactions between these points of view and describe several new strata in the moduli space $\overline{\mathfrak{M}}_{1,2}^{(Gor)}$. In particular, we succeed in describing explicitly such surfaces with normalisation, which is the symmetric product of an elliptic curve (see Section 4), a case that was left open in [FPR17].

The thesis is organised as the following: In Chapter 1 we introduce the main objects of interest and provide a translation between the description of the canonical ring and the structure equations for the quadruple cover. This translation becomes especially transparent for bi-double covers, that are, quadruple covers where the quotient map is induced by the action of $(\mathbb{Z}/2)^2$.

The structure equations for a quadruple cover lend themselves to a local study of its singularities, which we undertake in Chapter 2. While this approach did not yield the global classification results that we hoped for, it still gives some characterisations of the local geometry.

Normal Gorenstein stable surfaces were classified in [FPR15b] via Kodaira dimension and number of elliptic singularities. In Chapter 3, we study the classification of

Table 1: Overview over the cases

normal cases						
min. resolution	$\kappa(\tilde{X})$	$p_g(\tilde{X})$	$q(\tilde{X})$	**Elliptic sing.** $(d_1, .., d_r)$	Dim. of family	Reference
gen. type	2	1	0	(2)		
min. prop. ell.	1	1	1	(1)		Section 3.1.1
	1	0	0	(1)		Section 3.1.2
Enriques		1		(2)		Section 3.2
Abelian variety	0	0		(1,1)		Section 3.4
Bielliptic		0		(1,1)		Section 3.3
rational	1			(d)		Section 3.5
Ruled over ell. curve	0	1		(d_1, d_2)		Section 3.5

non-normal cases			
normalisation $\overline{X}$	(general) conductor	Dim. of family	Reference
$\mathbb{P}^2$	irred. quartic with 2 nodes		Section 4.2
del Pezzo of degree 2[a]	bi-elliptic curve in $\lvert -2K_{\overline{X}} \rvert$	10	Section 4.3
ruled surface over ell. curve	$D \in \lvert 2(C_0 + F) \rvert$ curve of genus 2		
symmetric square of elliptic curve	$D \in \lvert 3C_0 - F \rvert$ curve of genus 2		Section 4.4

[a] possibly with one elliptic singularity

Table 2: Overview over the non-normal cases

these strata in detail, adding many new cases and fleshing out the known ones.

Non-normal Gorenstein stable surfaces were also classified in [FPR15b]: their normalisation are either projective plane, a Del Pezzo surface, a ruled surface over an elliptic curve, or a symmetric product of an elliptic curve. For each case, we will study them in more detail. We also compute the canonical ring for each case and compare it to the algebraic translation in Chapter 1. This part belongs to Chapter 4. Some computer algebra computations are explained in Appendix A.

Acknowledgements. First of all, I would like to express my profound gratitude to my supervisor, Prof. Dr. Sönke Rollenske, whom I regard as my Ph.D. father. I am very grateful for his scientific guidance, patience, continued encouragement, and support throughout my Ph.D. research and my life in Germany. Secondly, I would like to thank my former colleagues, Dr. Andreas Krug and Dr. Ben Anthes, for their collaboration, patience, and advice. I appreciate the warm and productive atmosphere with them in our work, coffee breaks, and senior seminars.

I would further like to thank the German Academic Exchange Service (DAAD) for their financial support. Without their funding, I may not have a chance to work on this research project.

I would also like to thank Hans-Christian von Bothmer for his contributive comments to the thesis and his elegant assist in Macaulay2 computation.

Lastly, I wish to express my love and gratitude to my parent, my husband Hieu, and my daughter Hannah for their unconditional love and support. This dissertation is dedicated to them.

Einleitung

Es ist eine allgemeine Tatsache, dass die Modulräume von *schönen* Objekten in der algebraischen Geometrie, sagen wir glatte Varietäten, oft nicht kompakt sind. In der Regel gibt es jedoch eine modulare Kompaktifizierung, bei der die Grenzpunkte verwandten, aber komplizierteren Objekten entsprechen.

Seit langem ist eine solche modulare Kompaktifizierung für den Modulraum $\mathcal{M}_g$ der glatten Kurven von Geschlecht g bekannt. In [Mor] haben Kollár und Shepherd-Barron den ersten Schritt zur Konstruktion einer modularen Kompaktifizierung $\overline{\mathfrak{M}}$ für Giesekers modulierten Raum von (kanonischen Modellen von) Flächen der allgemeinen Typs [Gie77] studiert. Auch wenn sich die eigentliche Konstruktion des Modulraumes um mehrere Jahrzehnte verzögerte, weil gewaltige technische Hindernisse zu überwinden waren, war es von Anfang klar, dass die Objekten, die durch $\overline{\mathfrak{M}}$ parametrisiert werden sollen, Flächen mit halb-log-kanonischen Singularitäten und amplem kanonischen Divisor sind, kurz *stabile Flächen*. Heutzutage ist die Existenz der Kompaktifizierung bekannt und es ist gewinnbringend, einzlene Komponenten zu studieren, um ein Gefühl für die Geometrie stabiler Flächen zu bekommen.

In einer Serie von Artikeln [FPR15b, FPR15a, FPR17] haben Franciosi, Pardini und Rollenske herausgefunden, dass unter den zusätzlichen Annahmen, dass der kanonische Divisor Cartier ist, d.h., im Falle der Gorenstein stabile Flächen, die der kanonische Ring eine genaue Beschreibung liefern kann, insbesondere für kleine Invarianten.

Der Schwerpunkt des Dissertationsprojektes sind stabile Gorenstein-Flächen mit $K^2 = 1$ und $\chi = 2$. Klassischerweise wurde es herausgefunden, dass die kanonische Modelle glatter Flächen mit diesen Invarianten vierfache verzweigte Überlagerungen der projektiven Ebene sind [Cat80], und diese Beschreibung wird mit [FPR17] auf den stabilen Gorenstein Fall ausgedehnt.

Unsere Forschung von dieser Flächen wird daher von drei Blickpunkte geleitet: direkte geometrische Betrachtungen, wie sie in [FPR17] angestellt wurden, der Struktur des kanonischen Rings sowie der Struktur der vierfachen Überlagerung.

Für letzteres bauen wir auf den Artikel von Casnati und Ekedahl [CE96] auf, in dem die Theorie für Gorenstein-Überlagerungen von Grad höchstens 5 dargestellt wurde. Die Theorie zeigt, dass Gorenstein vierfache Überlagerungen der projektiven Ebene als Untervaritäten von Kodimension 2 in ein $\mathbb{P}^2$-Bündel $\mathbb{P}(\mathcal{E})$ eingebettet sind, lokal gegeben durch die Schnittmenge zweier relativer Quadriken.

Insgesamt arbeiten wir die Relationen und Wechselwirkungen zwischen diesen Standpunkten heraus und beschreiben mehrere neue Strata im Modulraum $\overline{\mathfrak{M}}_{1,2}^{(Gor)}$. Insbesondere können wir solche Flächen beschreiben, deren Normalisierung das symmetrische Produkt einer elliptischen Kurve ist (siehe Abschnitt 4), ein Fall, der in [FPR17] offen gelassen wurde.

Die Dissertation ist wie folgt organisiert: In Kapitel 1 stellen wir die Hauptobjekte unserer Betrachtungen vor. Wir geben eine Übersetzung zwischen der Beschreibung durch den kanonischen Ring und den Strukturgleichungen für die vierfache Abdeckung. Dies wird besonders transparent für Überlagerungen, bei denen die Quotientenabbildung durch eine Wirkung von $(\mathbb{Z}/2)^2$ induziert wird.

Die Strukturgleichungen der vierfachen Überlagerung bieten sich für eine lokalen Untersuchung ihrer Singularitäten, die wir in Kapitel 2 durchführen. Obwohl dieser

Zugang nicht die erhofften globalen Klassifikationsergebnisse erbracht hat, liefert er dennoch einige Charakterisierungen der lokalen Geometrie.

Normale stabile Gorenstein-Flächen wurden durch die Kodaira-Dimension und Anzahl der elliptischen Singularitäten in [FPR15b] klassifiziert. In Kapitel 3 untersuchen wir die Klassifikation dieser Strata im Detail. Dazu fügen wir viele neue Fälle und konkretisieren wir auch die bekannte Fälle.

Nicht-normale stabile Gorenstein-Flächen mit $K_X^2 = 1$ wurden ebenfalls in [FPR15b] klassifiziert: ihre Normalisierung ist entweder eine projektive Ebene, eine Del-Pezzo-Fläche, eine Regelfläche über einer elliptischen Kurve oder ein symmetrisches Produkt einer elliptischen Kurve. Für jeden Fall werden wir sie sowohl detailiert untersuchen als auch den kanonischen Ring berechnen damit vergleichen wir mit der algebraischen Übersetzung in Kapitel 1. Dieser Teil gehört zu Kapitel 4. Einige Computeralgebra-Berechnungen werden im Anhang A erläutert.

Danksagen. Zunächst möchte ich meinem Betreuer, Prof. Sönke Rollenske, den ich als meinen Doktorvater betrachte, meine besondere Anerkennung und meinen Dank für seine wertvolle Anleitung und Unterstützung während meiner Doktorarbeit und meines Lebens in Deutschland aussprechen. Zweitens möchte ich meinen ehemaligen Kollegen, Dr. Andreas Krug und Dr. Ben Anthes, für ihre Mitarbeit, ihre Geduld und ihre Anregungen danken. Ich schätze die warme und produktive Atmosphäre, in der wir während unserer Kaffeezeiten und Oberseminare mit ihnen zusammen waren.

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Zuletzt, aber am wichtigsten, möchte ich meinen Eltern, meinem Mann Hieu und meiner Tochter Hannah für ihre bedingungslose Liebe und Unterstützung danken. Diese Dissertation ist ihnen gewidmet.

Notations and conventions

We work with schemes of finite type over the complex numbers $\mathbb{C}$. A surface is a reduced, connected projective scheme of pure dimension two but not necessarily irreducible. A curve is a projective scheme of pure dimension one but not necessarily irreducible or connected. All schemes that we consider will be Cohen- Macaulay and thus admit a dualising sheaf ω_X.

Notations

Let X be a surface. We write

1. $p_g(X) = h^2(\mathcal{O}_X)$, and if C is a curve, then $p_a(C) = h^1(\mathcal{O}_X)$.
2. K_X = canonical divisor, which is a Weil-divisor whose support does not contain any component of the non-nonmal locus and such that $\mathcal{O}_X(K_X) \simeq \omega_X$.
3. Given an invertible sheaf $L \in \mathsf{Pic}(X)$, one defines the ring of sections

$$R(X, L) = \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mL));$$

   for $L = K_X$, we have the canonical ring $R(K_X) := R(X, K_X)$.
4. Let $\mathcal{E}$ be a vector bundle of rank n over X, we define a projective bundle $\mathbb{P}(\mathcal{E}) = \mathsf{Proj}(\mathsf{Sym}^* \mathcal{E}^\vee)$.

1. Algebraic and geometric models

In this section, the descriptions of a Gorenstein stable surface X with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$ by Franciosi, Pardini and Rollenske [FPR17] and also by Casnati, Ekedahl [CE96] are recalled. An explicit translation between them is then presented.

1.1. Stable surfaces

In this section, we are going to recall the definitions of *semi-log-canonical* (slc) surfaces from [Kol13, Section 5]. A demi-normal scheme is a finite type scheme X over $\mathbb{C}$ that satisfies the following conditions:

1. X satisfies the S_2 condition, i.e., for every $x \in X$ we have

$$\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}) \geq \min\{2, \dim(\mathcal{O}_{X,x})\}$$

2. At each point x of codimension one in X, x is either regular or is an ordinary double point.

We denote by $\pi \colon \overline{X} \to X$ the normalisation of X. The conductor ideal $\mathcal{I}_{\overline{D}} = \mathscr{H}om(\pi_*\mathcal{O}_{\overline{X}}, \mathcal{O}_X)$ is an ideal sheaf both in $\mathcal{O}_X$ and $\mathcal{O}_{\overline{X}}$ and as such defines subschemes $D \subset X$ and $\overline{D} \subset \overline{X}$, both reduced and of codimension one. We often refer to D as the non-normal locus of X.

Definition 1.1 — The demi-normal surface X is said to have *semi-log-canonical (slc)* singularities if it satisfies the following conditions:

1. The canonical divisor K_X is $\mathbb{Q}-$Cartier,

2. The pair $(\overline{X}, \overline{D})$ has log-canonical (lc) singularities.

It is called a stable surface if in addition K_X is ample. In that case we define the geometric genus of X to be $p_g(X) = h^0(X, \omega_X) = h^2(X, \mathcal{O}_X)$ and the irregularity as $q(X) = h^1(X, \omega_X) = h^1(X, \mathcal{O}_X)$. A Gorenstein stable surface is a stable surface such that K_X is ample.

1.2. Canonical ring in [FPR17]

The canonical ring of stable surfaces with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$ was computed in [FPR17]. We would like to remind the reader of the explicit computation. This method of computation finds its application in many cases in this thesis.

Lemma 1.2 — *[FPR17] Let X be a Gorenstein stable surface with $K_X^2 = 1$ and let $C \in |K_X|$ be a canonical curve.*

Then C is a reduced and irreducible Gorenstein curve with $p_a(C) = 2$, not contained in a non-normal locus.

Proof. When $C \in |K_X|$, then $\mathcal{O}_X(C) = \mathcal{O}_X(K_X) = \omega_X$ is an invertible sheaf on C, thus C is Gorenstein curve. Adjunction formula gives us $2g - 2 = \deg{(K + C)}|_C = C(C + K_C) = 2$, so $\chi(\mathcal{O}_C) = 1 - g = -1$ and $p_a(C) = 1 - \chi(\mathcal{O}_C) = 2$.

Since $K_X C = 1$ and K_X is an ample Cartier divisor, the curve C is reduced and irreducible. Indeed, by writing $C = \sum_{i=1}^{s} m_i C_i$, then $1 = K_X \cdot C = K_X \cdot \sum_{i=1}^{s} m_i C_i = \sum_{i=1}^{s} m_i K_X \cdot C_i$. In the last expression, we have $K_X \cdot C_i = \deg K_X|_{C_i} \geq 1$, because K_X is an ample Catier divisor. It follows that $1 \geq \sum_{i=1}^{s} m_i$, so $s = m_i = 1$.

Since no component of the non-normal locus is Cartier and C is reduced, C can not be contained in the non-normal locus. □

Let C be an irreducible Gorenstein curve of genus 2 and let $L \in \mathrm{Pic}(C)$ be a square root of K_C. In our application, C is a canonical curve and $L = K_X|_C$.

We denote by $\overline{S_2}$ the polynomial ring $\mathbb{C}[y_1, y_2, z_1, z_2]$ where y_i has degree 2 and z_i has degree 3 ($i = 1, 2$).

Proposition 1.3 — *[FPR17] Let C be an integral Gorenstein curve with $p_a(C) = 2$ and let $L \in Pic(C)$ such that $L^{\otimes 2} = \omega_C$. If $h^0(L) = 0$, then $R(L) \cong \overline{S_2}/(f_1, f_2)$, where $f_1 = z_1^2 + c_1(y_1, y_2)$ and $f_2 = z_2^2 + c_2(y_1, y_2)$ are weighted homogeneous of degree 6 and c_1, c_2 have no common factor.*

Proof. Two main tools used to prove this Proposition are

1. the Riemann-Roch theorem and Serre's duality, used to compute $h^0(mL), m \geq 1$ and to determine the base points of $|mL|$;
2. the base point pencil trick, used to show surjectivity of multiplication maps of the form $H^0(aL) \otimes H^0(bL) \to H^0((a+b)L)$.

1. Compute $h^0(mL)$: First we have the Riemann-Roch formular for mL:

$$h^0(mL) - h^0(K_C - mL) = \deg mL + 1 - g \tag{1.4}$$

where:

- $h^0(K_C - mL) = 0$ for $m \geq 3$, since $K_X - 3L = -L$ has no global section,
- $\deg(mL) = m \deg(L) = m$,
- $\deg(K_X|_C) = K_X^2 = 1$ $(C \in |K_X|)$,
- $g = p_a(C) = 2$.

Then we get

$$h^0(mL) = \begin{cases} 0 \text{ or } 1 & \text{if } m = 1 (\text{depends on} L) \\ 2 & \text{if } m = 2 \\ m - 1 & \text{if } m \geq 2. \end{cases}$$

2. Compute the canonical ring $R(L)$:

$$\begin{aligned} R(L) &= \oplus_{m \geq 0} H^0(mL) \\ &= \mathbb{C} \oplus R_1 \oplus R_2 \oplus R_3 + \dots \end{aligned}$$

where

- $R_0 \cong \mathbb{C}$ and $R_0 = \langle 1 \rangle$ has dimension 1.
- $R_1 = H^0(L)$, and we assume that it has dimension $h^0(L) = 0$.
- $R_2 = H^0(2L)$ has dimension $h^0(2L) = h^0(K_X) = 2$. We have that $C \in |K_X|$ has no base point, because otherwise we have $h^0(K_C - P) = h^0(K_C) = 2$ for P a base point of $|K_C|$, but $\deg(K_C - P) = \deg K_C - \deg P = 1$ which is contradiction because $C \ncong \mathbb{P}^1$.

  With $|K_C|$ base point free we could associate a morphism $\pi : C \overset{|K_C|}{\to} \mathbb{P}^1$, thus $R_2 = \langle y_1, y_2 \rangle$
- $R_3 = \langle z_1, z_2 \rangle$
- $R_4 = \langle y_1^2, y_1y_2, y_2^2 \rangle$
- $R_5 = \langle y_1z_1, y_2z_2, y_1z_2, y_2z_1 \rangle$
- $R_6 = \langle y^3, y_1^2y_2, y_1y_2^2, y_2^3 \rangle + \langle z_1^2, z_1z_2, z_2^2 \rangle$ brings the two relations in the forms: $z_1^2 + c_1(y_1, y_2)$ and $z_2^2 + c_2(y_1, y_2)$

At this point, we still need to show that these are only relations in $\overline{S_2}$. Therefore we use the base point free pencil trick: For $a \geq 2$ let x, y be elements without common zeros in $H^0(aL)$, then we get a map $\pi : X \to \mathbb{P}^1$ so, that $aL = \pi^*(\mathcal{O}(1))$. Consider the usual restriction sequence on $\mathbb{P}^1$:

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1) \to 0$$

by pullback on X and tensor with bL we get

$$0 \to (b-a)L \to \langle x, y \rangle \otimes bL \to (a+b)L \to 0$$

Thus

$$0 \to H^0((b-a)L) \to \langle x, y \rangle \otimes H^0(bL) \to H^0((a+b)L) \to H^1((b-a)L)$$

where $\langle x, y \rangle \otimes H^0(bL) \subset H^0(aL) \otimes H^0(bL)$. To control the surjectivity of the map $H^0(aL) \otimes H^0(bL) \to H^0((a+b)L)$ we need to control $H^1((b-a)L)$. Here $a = 2$, then $H^1((b-2)L) = 0$ if and only if $b \geq 5$. There are only two relations up to degree 6. WLOG we can assume that c_1 and c_2 have common divisor y_1. Then the point $A = (0 : 1 : 0 : 0)$ is a singular point of the curve and a base point of the 1-dimensional system $|3L|$. It follows that A is double point of C, the fixed part of $|3L|$ is equal to $|2A|$ and the moving part M is a linear system of dimension 1 and degree 1, contradict the assumption that C has genus 2. □

We denote by S the polynomial ring $\mathbb{C}[x_0, y_1, y_2, z_1, z_2]$ where x_0 has degree 1, y_i has degree 2 and z_i has degree 3, $i = 1, 2$.

Theorem 1.5 — *[FPR17] Let X be a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(X) = 2$, then $q(X) = 0$ and $R(K_X) \cong S/(f_1, f_2)$, where*

$$
\begin{aligned}
f_1 &= z_1^2 + z_2 x_0 a_1(x_0, y_1, y_2) + b_1(x_0, y_1, y_2) \\
f_2 &= z_2^2 + z_1 x_0 a_2(x_0, y_1, y_2) + b_2(x_0, y_1, y_2)
\end{aligned}
\tag{1.6}
$$

are weighted homogeneous of degree 6. Hence X can be canonically embedded as a complete intersection of bidegree $(6, 6)$ in (the smooth locus of) $\mathbb{P}(1, 2, 2, 3, 3)$.

Proof. We have $q = 0$ by Theorem 2.2 in [FPR17] and also $p_g(X) = 1$. Now let $C \in |K_X|$ and set $L = K_X|_C$ so that by adjunction we have $\mathcal{O}_C(K_C) = \omega_C = \omega_X \otimes \mathcal{O}_C(C)|_C = \mathcal{O}_X(C) \otimes \mathcal{O}_C(C)|_C = \mathcal{O}_C(C)^{\otimes 2} = L^{\otimes 2}$, and let $x_0 \in R(K_X)$ be a section defining C. The pair (C, L) satisfies the hypothesis of Proposition 1.3. Consider the usual restriction sequence

$$
0 \to \mathcal{O}(mK_X - C) \xrightarrow{\cdot x_0} \mathcal{O}(mK_X) \to L^{\otimes m} \to 0
$$

Since $q(X) = 0$ and $H^1(mK_X) = 0$ for $m \geq 2$ by the Kodaira Vanishing Theorem, we see that the map $R(K_X)/x_0 \to R(L)$ is a surjection, and hence an isomorphism. In particular, $h^0(L) = p_g(X) - 1 = 0$, so that this case corresponds to the case of Proposition 1.3. The claim about generators and relations is now obtained by lifting the relations of $R(L)$ to $R(K_X)$ and completing the squares in the lifted equations.

For this case, when $p_g = 1$: the singular locus of $\mathbb{P}(1, 2, 2, 3, 3)$ is the union of two lines $\mathbb{P}(2, 2)$ and $\mathbb{P}(3, 3)$, which do not meet X in view of format of the equations since the polynomials $b_1(x_0, y_1, y_2)$ and $b_2(x_0, y_1, y_2)$ have no common factors by Proposition 1.3. □

Corollary 1.7 — *The bi-canonical map $\varphi : X \xrightarrow{|2K_X|} \mathbb{P}^2$ is a quadruple cover.*

Proof. It follows from the two equations of X in the previous theorem that the map $|2K_X| : X \to \mathbb{P}^2$ is induced from the inclusion of rings $\mathbb{C}[x^2, y_1, y_2] \hookrightarrow R(K_X)$. □

Proposition 1.8 — *Let X be a Gorenstein stable surface with $K_X^2 = 1$ and $\chi = 2$. Then*

1) $\mathrm{Aut}(X/\mathbb{P}^2)$ *is one of groups* $0, \mathbb{Z}/2, (\mathbb{Z}/2)^2$.

2) $\mathrm{Aut}(X/\mathbb{P}^2) = (\mathbb{Z}/2)^2$ *if and only if one can choose* $a_1 = a_2 = 0$ *in the equations* (1.6).

3) $\mathrm{Aut}(X/\mathbb{P}^2) = \mathbb{Z}/2$ *if and only if only one can choose one of the* $a_i = 0$ *but not both.*

Proof. First of all note that in case $a_1 = 0$ in the coordinates as in (1.6), then $(x, y_1, y_2, z_1, z_2) \mapsto (x, y_1, y_2, -z_1, z_2)$ defines a $\mathbb{Z}/2$-action on the canonical ring and on X over $\mathbb{P}^2$, and similarily for $a_2 = 0$. So it remains to exclude the case $\mathbb{Z}/4$ and to prove the "only if" part.

Let G be a non-trivial group acting on X over $\mathbb{P}^2$, so that we have a factorisation $X \to X/G \to \mathbb{P}^2$. Since the bi-canonical map is of degree 4, we see that the order of G is 2 or 4. The action of G on X induces an action on the canoncial ring R, which leaves R_2 invariant, that is $\mathbb{C}[x^2, y_1, y_2] \subset R^G$.

Our arguments now rely on the examination of the decompositions of R_3 and R_6 into G-invariant subspaces:

$$
\begin{gathered}
R_1 = \langle x \rangle, \quad R_1^{\otimes 2} \text{ trivial} \\
R_2 = R_1^{\otimes 2} \oplus \langle y_1, y_2 \rangle \\
R_3 = R_1^{\otimes 3} \oplus R_1 \otimes \langle y_1, y_2 \rangle \oplus U, \quad U \text{ possibly reducible but effective} \\
R_6 = S^2 U \oplus R_1 \otimes U \oplus S^3 U / \langle f_2, f_2 \rangle
\end{gathered}
$$

Since the relations f_i are of the form given in Theorem 1.5, we see that they both contain a non-trivial G-invariant summand and thus are in the subspace R_6^G. Furthermore, applying the projection $S^2 U \oplus R_1 \otimes U \oplus S^3 R_3$ to $\langle f_1, f_2 \rangle$ gives a 2-dimensional G-invariant subspace of $S^2(U)$, spanned by z_1^2, z_2^2 in the coordinates chosen in Theorem 1.5.

It is easy to see that for an effective, 2-dimensional $\mathbb{Z}/4$-representation U this cannot happen, so $G \neq \mathbb{Z}/4$.

Let $\sigma \in G$ be a non-trivial element with neccesarily $\sigma^2 = 1$. Let z_1, z_2 be eigenvectors for the action of σ on U. Then

$$
\begin{aligned}
\sigma : \mathbb{C}[x, y_1, y_2, z_1, z_2] &\to \mathbb{C}[x, y_1, y_2, z_1, z_2] \\
(x, y_1, y_2, z_1, z_2) &\mapsto v = (\pm x, y_1, y_2, \pm z_1, \pm z_2)
\end{aligned}
$$

Note that the action on X is nontrivial and that, up to the (weighed) $\mathbb{C}^*$-action and renumbering, we have the following 3 cases to consider:

- $v = (x, y_1, y_2, z_1, z_2)$. In this case σ acts trivially on X which was excluded.
- $v = (-x, y_1, y_2, -z_1, z_2)$. In this case, the only invariant monomials of degree 6 involving z_i are z_1^2, z_2^2, xz_1, so we have relations of the form

  $$z_1^2 + 2\tilde{a}_1 x z_1 + \tilde{b}_1 = (z_1 + x a_1)^2 + \tilde{b}_1', \quad z_2^2 + \tilde{a}_2 x z_1 + \tilde{b}_2,$$

  which after a coordinate change give the desired equation with $a_1 = 0$.
- $v = (-x, y_1, y_2, z_1, z_2)$. In this case, the only invariant monomials of degree 6 involving z_i are $z_1^2, z_1 z_2, z_2^2$, so we have relations of the form

  $$z_1^2 + b_1, \quad z_2^2 + b_2,$$

  after possibly a linear change of coordinates in the subspace U.

To conclude we only have to note that in the case of an effective $(\mathbb{Z}/2)^2$-action on X, we will neccesarily have an involution which acts as in the last case. $\square$

1.3. Quadruple covers after [CE96]

Let X be a Gorenstein stable surface satisfying $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$. It is proven in the previous section that X is a quadruple cover of the projective plane $\mathbb{P}^2$. We can apply the general structure theorem of Gorenstein covers of degree d, $(d \geq 3)$ described in [CE96]. To any quadruple cover $\varrho\colon X \to \mathbb{P}^2$, one can associate the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{\varrho^\#} \varrho_*\mathcal{O}_X \longrightarrow \mathcal{E}^\vee \longrightarrow 0, \tag{1.9}$$

where $\varrho_*\mathcal{O}_X$ is a locally free sheaf of rank 4 and $\mathcal{E}^\vee \cong \mathsf{coker}\,\varrho^\#$ is a locally free sheaf of rank 3. The embedding $\varrho^\#\colon \mathcal{O}_Y \to \varrho_*\mathcal{O}_X$ admits a splitting $\frac{1}{4}\mathrm{tr}\colon \varrho_*\mathcal{O}_X \to \mathcal{O}_Y$, i.e., $\varrho_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{E}^\vee$.

As an example to understand the above trace map, we replace $\mathbb{P}^2$ by $\mathsf{Spec}(\mathbb{C})$. $\varrho_*\mathcal{O}_X$ is then a vector space of dimension 4 over $\mathbb{C}$. Multiplication by an element x induces an endomorphism $\cdot x\colon \varrho_*\mathcal{O}_X \to \varrho_*\mathcal{O}_X$. Taking into account that $\mathrm{tr}(1_{\varrho_*\mathcal{O}_X}) = \dim_{\mathbb{C}}(\varrho_*\mathcal{O}_X) = 4$, the trace map $\mathrm{tr}\colon \varrho_*\mathcal{O}_X \to \mathbb{C}$ is defined as the trace of this endomorphism, namely $\mathrm{tr}(x) = \mathrm{tr}(\cdot\colon \varrho_*\mathcal{O}_X \to \varrho_*\mathcal{O}_X)$. So $\frac{1}{4}\mathrm{tr}$ splits the inclusion $\mathcal{O}_Y \to \varrho_*\mathcal{O}_X$.

Although [CE96] described general results for covers of degree $d \geq 3$, we restrict our attention to some important points of the theorem which can be applied to our situation. Let $\varrho\colon X \to \mathbb{P}^2$ be a Gorenstein quadruple cover of the projective plane, where X is a Gorenstein stable surface satisfying $K^2 = 1$ and $\chi(\mathcal{O}_X) = 2$. Locally, each fibre $X_y, y \in \mathbb{P}^2$ is a zero-dimensional Gorenstein scheme of length 4 which is a complete intersection of two conics. Therefore, the minimal free resolution of the homogeneous coordinate ring S_{X_y} over $S = \mathbb{C}[x, z_1, z_2]$ is

$$0 \to S(-4) \to S(-2)^{\oplus 2} \to S \to S_{X_y} \to 0.$$

The minimal resolution of $\mathcal{O}_{X_y}$ is the sheafification of the previous exact sequence and hence is

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{X_y} \to 0, \tag{1.10}$$

where the map σ defines X_y as a complete intersection of two conics in $\mathbb{P}^2$.

Globally, in [CE96] it is shown that X is embedded in the projective $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}^2$ such that $\varrho = \iota \circ \pi$ and there is, uniquely up to unique isomorphism, an exact sequence

$$0 \to \pi^* \det \mathcal{E}(-4) \to \pi^*\mathcal{F}(-2) \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_X \to 0 \tag{1.11}$$

whose restriction on each fiber $\mathbb{P}_y = \pi^{-1}(y)$ over y is a minimal free resolution of the structure sheaves over $X_y = \varrho^{-1}(y)$ as in the sequence (1.10) The idea to contruct the exact sequence (1.11) can be seen as the following diagram, where the main tools used to build this sequence are the Theorem of Grauert [Har77, Chapter III Corrolary 12.9].

$$
\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathcal{A}_2 \arrow[r] & \mathcal{N}(-4) \arrow[r] \arrow[dr] & \mathcal{A}_1 \arrow[r] \arrow[d] & 0 & & \\
& & & \pi^*\mathcal{F}(-2) \arrow[d] \arrow[dr] & & & \\
& & 0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
& & & 0 & & &
\end{tikzcd}
$$

where $\mathcal{F} = \pi_*\mathcal{I}(2)$ and $\alpha\colon \pi^*\mathcal{F}(-2) \to \mathcal{I}$ is the natural evaluation map whose restriction coincides with the map $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \to \mathrm{Im}(\sigma)$ and thus is surjective. Let $\mathcal{A}_1 = \ker(\alpha)$ and we apply the same proceed for this sequence, in this way we get the sequence

$$0 \to \mathcal{N}(-4) \to \pi^*\mathcal{F}(-2) \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_X \to 0 \tag{1.12}$$

The map $\mathcal{N}(-4) \to \pi^*\mathcal{F}(-2)$ is injective by the Lemma of Nakayama. Now we compare the sequence (1.12) with the Koszul complex of σ, which is fibrewise (hence globally) exact since $\dim(X_y) = 0$, taking into account the uniqueness of (1.12) we get $\det \mathcal{N} \cong \pi^* \det \mathcal{F} \cong \pi^* \det \mathcal{E}$, thus $\det \mathcal{F} \cong \det \mathcal{E}$ since π^* is injective. In this way, we obtain the exact sequence (1.11).

The section $\eta \in H^0(\mathbb{P}^2, \mathcal{S}^2\mathcal{E} \otimes \check{\mathcal{F}})$ comes from the natural isomorphism

$$\Phi\colon H^0(\mathbb{P}^2, \mathcal{S}^2\mathcal{E} \otimes \check{\mathcal{F}}) \to H^0(\mathbb{P}^2, \pi^*\mathcal{F}(-2)).$$

Moreover, in [CE96] it is shown that the embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$ is induced from the surjective morphism $\varphi\colon \varrho^*\mathcal{E} \to \varrho^*\varrho_*\omega_{X/\mathbb{P}^2} \to \omega_{X/\mathbb{P}^2}$ and the ramification divisor R satisfies $\mathcal{O}_X(R) \cong \omega_{X/\mathbb{P}^2} \cong \mathcal{O}_X(1) := \iota^*\mathcal{O}_{\mathbb{P}}(1)$.

Remark 1.13 — In [CE96], the quadruple cover of $X \to \mathbb{P}^2_{\mathbb{C}}$ was proven where X is a canonical model of a minimal surface S with $K_S^2 = 1$ and $\chi(\mathcal{O}_S) = 2$. The arguments remain correct for Gorenstein stable surface.

We will reprove in more details in the following:

Proposition 1.14 — *Let X be a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$, let $\varrho\colon X \to \mathbb{P}^2$ be a quadruple cover of $\mathbb{P}^2$. Then*

1. *There exists an embedding $\iota\colon X \hookrightarrow \mathbb{P}(\mathcal{E})$ so that the following diagram commutes*

$$
\begin{tikzcd}
X \arrow[r, hook, "\iota"] \arrow[d, "\varrho"] & \mathbb{P}(\mathcal{E}) \arrow[dl, "\pi"] \\
\mathbb{P}^2 &
\end{tikzcd}
$$

2. *The locally free sheaf $\mathcal{E}$ has the presentation $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$.*

3. *The vector bundle $\mathcal{F}$ in the building data of the quadruple cover is such that $\mathcal{F} \cong \mathcal{F}^\vee(7)$ sits inside the following short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(4) \oplus \mathcal{O}_{\mathbb{P}^2}(6) \to \mathcal{O}_{\mathbb{P}^2}(7) \to 0$$

Proof. First we compute the dualizing sheaf $\omega_{X|\mathbb{P}^2}$ and its push forward by using the relative duality in [Kle80]. By using $\varrho^*\mathcal{O}_{\mathbb{P}^2}(1) = \omega_X^{\otimes 2} = \mathcal{O}_X(2K_X)$, we have

$$\begin{aligned}\omega_{X|\mathbb{P}^2} &= \mathcal{O}_X(K_X - \varrho^* K_{\mathbb{P}^2}) \\ &= \mathcal{O}_X(K_X + 6K_X) \\ &= \mathcal{O}_X(7K_X)\end{aligned}$$

We apply the relative duality in [Kle80] to write $\omega_{X|\mathbb{P}^2} = \mathscr{H}\mathrm{om}(\mathcal{O}_X, \omega_{X|\mathbb{P}^2})$. Then

$$\begin{aligned}\varrho_*\omega_{X|\mathbb{P}^2} &= \mathscr{H}\mathrm{om}_X(\varrho_*\mathcal{O}_X, \varrho^*\mathcal{O}_{\mathbb{P}^2} \otimes \omega_{X|\mathbb{P}^2}) \\ &= \mathscr{H}\mathrm{om}_{\mathbb{P}^2}(\varrho_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^2}) \\ &= \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}\end{aligned}$$

Let $\mathcal{E}$ be a locally free sheaf of rank 3 on $\mathbb{P}^2$. Then by [Har77, Proposition 7.12], to give a morphism of X to $\mathbb{P}(\mathcal{E})$ over $\mathbb{P}^2$, it is equivalent to give an invertible sheaf $\mathcal{L}$ on $\mathbb{P}^2$ and a surjective morphism on $\mathbb{P}^2$, $\varrho^*\mathcal{E} \to \mathcal{L} = \iota^*\mathcal{O}_{\mathbb{P}}(1)$. For $\mathcal{L} \cong \omega_{X|\mathbb{P}^2}$ we have

$$\begin{aligned}\mathsf{Hom}(\varrho^*\mathcal{E}, \mathcal{L}) &\cong \mathsf{Hom}(\mathcal{E}, \varrho_*\omega_{X|\mathbb{P}^2}) \\ &\cong \mathsf{Hom}(\mathcal{E}, \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E})\end{aligned}$$

and we can consider the natural inclusion $\mathcal{E} \hookrightarrow \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}$. This induces a surjective map $\varrho^*\mathcal{E} \twoheadrightarrow \omega_{X|\mathbb{P}^2}$, because locally, we can write $\mathcal{E} = \mathcal{O}_X^{\oplus 3}$ and the surjectivity is equivalent to images of sections in $\mathcal{E}$ that generate $\mathcal{L}$. For the second statement we note from (1.9) that $\varrho_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}^\vee$, we wish to compute the cohomology $h^1(\mathbb{P}^2, \mathcal{E}^\vee(t))$ and use the Horrocks' Theorem [OSSG80, Theorem 2.3.1]. For any $t \in \mathbb{Z}$ we have

$$h^1(\mathbb{P}^2, \mathcal{E}^\vee(t)) = h^1(X, \varrho_*\mathcal{O}_X(t)) = h^1(\varrho^*\mathcal{O}_{\mathbb{P}^2}(t)) = h^1(X, \omega_X^{2t}) = 0$$

Then by Horrocks' Theorem, $\mathcal{E}$ splits as direct sum of invertible sheaves, $\mathcal{E} \cong \oplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^2}(\alpha_i)$, and for $t \geq 2$ we get

$$\frac{2t(2t-1)}{2} + 2 = h^0(X, \varrho^*\mathcal{O}_{\mathbb{P}^2}(t)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)) + h^0(\mathbb{P}^2, \mathcal{E}^\vee(t))$$

equivalently,

$$2t^2 - t + 2 = \binom{t+2}{2} + \binom{\alpha_1 + t + 2}{2} + \binom{\alpha_2 + t + 2}{2} + \binom{\alpha_3 + t + 2}{2}.$$

We compare the coefficients of two polynomials on two sides of this equation and get $\alpha_1 + \alpha_2 + \alpha_3 = -7$, $\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 = 17$ with $\alpha_i \in \mathbb{Z}$. This implies $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 3)$. Thus $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$.

The statement $\mathcal{F} \cong \mathcal{F}^\vee(7)$ follows from [CE96] that the exact sequence (1.11) is self-dual. Indeed, the dual of Sequence (1.11) is

$$0 \to \pi^* \det \mathcal{E}(-4) \to \pi^*\mathcal{F}^\vee(5) \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0$$

it follows that $\pi^*\mathcal{F}(-2) \cong \pi^*\mathcal{F}^\vee(5)$, i.e $\mathcal{F} \cong \mathcal{F}^\vee(7)$ since π^* is injective. Moreover, the locally free sheaf $\mathcal{F}$ of rank 2 is push forward of the ideal sheaf $\mathcal{I}$. Indeed, by projecting the following exact sequence

$$0 \to \mathcal{I}(2) \to \mathcal{O}_{\mathbb{P}}(2) \to \mathcal{O}_X(2) \to 0,$$

and taking into account that $R^1\pi_*\mathcal{I}(2) = 0$, setting $\mathcal{F} := \pi_*\mathcal{I}(2)$, we obtain the following exact sequence for $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{S}^2\mathcal{E} \to \varrho_*\omega^2_{X|\mathbb{P}^2} \to 0.$$

Here, $\varrho_*\omega^2_{X|\mathbb{P}^2}$ can be computed using dualizing as in [Kle80]

$$\begin{aligned}\varrho_*\omega^2_{X|\mathbb{P}^2} &= \varrho_*\varrho^*\mathcal{O}_{\mathbb{P}^2}(7)\\ &= \varrho_*(\mathcal{O}_X \otimes \varrho^*\mathcal{O}_{\mathbb{P}^2}(7))\\ &= \varrho_*\mathcal{O}_X(7) = (\mathcal{O}_{\mathbb{P}^2} \oplus \check{\mathcal{E}})(7).\end{aligned}$$

which leads to the this exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^2}(4)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(5)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(6) \to \mathcal{O}_{\mathbb{P}^2}(4) \oplus \mathcal{O}_{\mathbb{P}^2}(5)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(7) \to 0$$

in addition $\mathcal{F}$ is stable vector bundle by [CE96].

Also by [CE96], the scheme $Z(s)$ of each $s \in H^0(\mathbb{P}^2, \mathcal{F}(-3))$ is non empty since $c_2(\mathcal{F}(-3)) = 3$ and has codimension 2 (otherwise $h^0(\mathbb{P}^2, \mathcal{F}(-4)) \neq 0$), it yields

$$0 \to \mathcal{F} \xrightarrow{.s} \mathcal{F}(-3) \to \mathcal{I}_Z(1) \to 0$$

Moreover, Z is the complete intersection of a line and a cubic which fits into this sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{I}_Z(1) \to 0$$

The following diagram describes the exactness of the short sequence of $\mathcal{F}$ in the last statement.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{\mathbb{P}^2}(-3) \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}(-3) \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2} \arrow[r, ".s"] & \mathcal{F}(-3) \arrow[r] \arrow[d] & \mathcal{I}_Z(1) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

$\square$

Combining these considerations with the results from [CE96] we get the following result.

Theorem 1.15 — *Let X be a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$. Then the bi-canonical map $\varrho\colon X \to \mathbb{P}^2$ is a Gorenstein cover of degree 4 whose invariants are $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$ and $\mathcal{F}$, where $\mathcal{F}$ is a locally free stable sheaf of rank 2 fitting into the exact sequence*

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(4) \oplus \mathcal{O}_{\mathbb{P}^2}(6) \to \mathcal{O}_{\mathbb{P}^2}(7) \to 0.$$

Conversely, given a locally free $\mathcal{O}_{\mathbb{P}^2}-$sheaf $\mathcal{F}$ of rank 2, a locally free sheaf $\mathcal{E}$ of rank 3 with $\det \mathcal{F} = \det \mathcal{E}$ and $\eta \in H^0(\mathbb{P}^2, \check{\mathcal{F}} \otimes S^2\mathcal{E})$ such that zero locus of η at every $y \in \mathbb{P}^2$ is a zero dimensional subscheme of length 4 defines a Gorenstein quadruple cover $\varrho\colon X \to \mathbb{P}^2$ such that $\check{\mathcal{E}} \cong coker \varrho^{\#}$ and $\mathcal{F} \cong ker(S^2\mathcal{E} \to \varrho_*\omega^2_{X|Y})$.

Remark 1.16 — This theorem give us a geometric model of X via describing the building data of the quadruple cover $X \to \mathbb{P}^2$. Another geometric model of X is described in Theorem 1.5, where X is complete intersection of bidegree $(6,6)$ in $\mathbb{P}(1,2,2,3,3)$. Moreover, the algebraic model of X is given by the two equations (1.6).

To connect the two algebraic models, we need some preparations which are going to be discussed in the following section.

1.4. The bi-canonical map and bi-canonical ring of surface

Let X be a Gorenstein stable surface with $K_X^2 = 1$ and $p_g(X) = 1$. Then according to [FPR17, Section 1], the map $\varrho\colon X \to \mathbb{P}^2$ is a finite morphism of degree 4 and thus a quadruple cover of $\mathbb{P}^2$ via the bicanonical map. It follows from Theorem 1.5 that X is canonically embedded as a complete intersection of degree $(6,6)$ in $\mathbb{P}(1,2,2,3,3)$ by computing the canonical ring of X. A computation of the bi-canonical ring of X is expressed as in the following:

We use the notations of Theorem 1.5. Recall in particular that $S = \mathbb{C}[x, y_1, y_2, z_1, z_2]$ with weights $(1,2,2,3,3)$.

Proposition 1.17 — *Let X be a Gorenstein stable surface with $K^2 = 1$ and $\chi(\mathcal{O}_X) = 2$.*

1. *The Veronese subring $S^{[2]} \subset S$ is generated by $u = x^2, y_1, y_2, v_1 = xz_1, v_2 = xz_2, w = z_1z_2, z_1^2, z_2^2$*

2. *The map $S^{[2]} \to R(K_X)^{[2]}$ is surjective with kernel generated by the ideal $J =$*

$(f_1, ..., f_8)$, *where*

$$f_1 = uw - v_1v_2$$
$$f_2 = u(v_2a_1 + b_1) + v_1^2$$
$$f_3 = u(v_1a_2 + b_2) + v_2^2$$
$$f_4 = v_2(v_2a_1 + b_1) + v_1w$$
$$f_5 = v_1(v_1a_2 + b_2) + v_2w$$
$$f_6 = (v_2a_1 + b_1)(v_1a_2 + b_2) - w^2$$
$$f_7 = z_1^2 + v_2a_1 + b_1$$
$$f_8 = z_2^2 + v_1a_2 + b_2$$

After eliminating the generators z_1^2, z_2^2 *via the relations* f_7, f_8 *we get*

$$R(K_X)^{[2]} \cong \mathbb{C}[u, y_1, y_2, v_1, v_2, w]/(f_1, \ldots, f_6).$$

Proof. For computation we will denote $V = \langle u, y_1, y_2\rangle$, where $u := x^2$. Let $v_1 := xz_1, v_2 := xz_2, w := z_1z_2$. The computation and relations are shown in Table 3 below.

Table 3: Bi-canonical ring

m	$\mathbf{h^0(mK_X)}$	$\mathbf{H^0(mK_X)}$	**Relations**
2	3	V	No relations
4	8	S^2V, v_1, v_2	No relations
6	17	S^3V $\langle v_1, v_2\rangle V$ $\langle z_1^2, z_1z_2, z_2^2\rangle$	$z_1^2 + v_2a_1 + b_1$ $z_2^2 + v_1a_2 + b_2$
8	30	S^4V $\langle xz_1, xz_2\rangle S^2V$ $\langle z_1^2, w, z_2^2\rangle V$	$wu = v_1v_2$ $v_1^2 + u(a_1v_2 + b_1)$ $v_2^2 + u(a_2v_1 + b_2)$
10	37	S^5V $\langle xz_1, xz_2\rangle S^3V$ $\langle z_1^2, z_1z_2, z_2^2\rangle S^2V$ $\langle z_1^3, z_1^2z_2, z_1z_2^2, z_2^3\rangle x$	$v_1w + v_2(a_1v_2 + b_1)$ $v_2w + v_1(a_2v_1 + b_2)$
12	68	S^6V $\langle xz_1, xz_2\rangle S^4V$ $\langle z_1^2, z_1z_2, z_2^2\rangle S^3V$ $\langle z_1^3, z_1^2z_2, z_1z_2^2, z_2^3\rangle xV$ $\langle z_1^4, z_1^3z_2, z_1^2z_2^2, z_1z_2^3, z_2^4\rangle$	$w^2 - (a_1v_2 + b_1)(a_2v_1 + b_2)$

The map $S^{[2]} \to R(K_X)^{[2]}$ is surjective with kernel J generated by eight relations as in Table 3. After eliminating the generators z_1^2, z_2^2 via the relations f_7, f_8 we get

$$R(K_X)^{[2]} \cong \mathbb{C}[u, y_1, y_2, v_1, v_2, w]/(f_1, \ldots, f_6),$$

where for each i, the polynomial f_i is weighted homogeneous of the appropriate degree in $\mathbb{C}[u, y_1, y_2, v_1, v_2, w]$. $\square$

1.5. Algebraic translation

In this section, we would like to investigate the relationship between two algebraic models which are considered as a finite morphism of degree 4, $\varrho\colon X \to \mathbb{P}^2$. As we computed in Proposition 1.17, X is embedded in $\mathbb{P}[2,2,2,4,4,6]$ and the bicanonical map $|2K_X|\colon X \to \mathbb{P}^2$ is a finite morphism of degree 4, which is induced from the natural map $\mathbb{C}[u, y_1, y_2] \hookrightarrow R(K_X)^{[2]}$. Moreover, it is shown in Section 1.3 that the datum of a Gorenstein quadruple cover ϱ is equivalent to the datum of a locally free sheaf $\mathcal{E}$ of rank 3, a locally free sheaf $\mathcal{F}$ of rank 2 and a section $\eta \in H^0(\mathcal{E}^2 \otimes \check{\mathcal{F}})$. The following diagram sumarizes these two descriptions:

$$\begin{tikzcd}
\mathbb{P} = \mathbb{P}(\mathcal{E}) = \mathsf{Proj}(\mathsf{Sym}\,\mathcal{E}) \arrow[dr, "\pi"'] & X \arrow[l, hook', "\iota"'] \arrow[r, hook, "\alpha"] \arrow[d, "\varrho"] & \mathbb{P}[2,2,2,4,4,6] \arrow[dl, dashed] \\
& \mathbb{P}^2 &
\end{tikzcd}.$$

We fix notations $R = \mathbb{C}[x^2, y_1, y_2]$ and $S = \mathbb{C}[x, y_1, y_2, z_1, z_2]$ as the polynomial rings where x has degree 1, y_i has degree 2 and z_i has degree 3. Our theorem provides a computation of the R-module F with respect to the vector bundle $\mathcal{F}$ of rank 2 and local equation of X in $\mathbb{P}^2 \times \mathbb{A}^2$ by using the equations of complete intersection in [FPR15b]. In addition to local equations, we describe how to extract the data of the quadruple cover from the description of the (bi)-canonical ring. Over suitable affine subsets of $U \subset \mathbb{P}^2$, we show that $\varrho^{-1}(U)$ is the intersection of two (explicitly given) conics in $U \times \mathbb{P}^2$.

Proposition 1.18 — *Let X be a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$. Let $\varrho\colon X \to \mathbb{P}^2$ be the quadruple cover as in the above description. Then the locally free sheaf $\mathcal{E}$ is the sheaf associated to the graded R-module $E = z_1R(2) \oplus z_2R(2) \oplus xR(3)$ in the bi-canonical ring of X.*

Proof. The computation of the module E is based on the computation of the canonical ring of the surface X. According to Section 1.3, $\omega_{X|\mathbb{P}^2} = \mathcal{O}_X(7K_X)$, $\varrho_*\omega_{X|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}$ and we have $H^0(\varrho_*\omega_{X|\mathbb{P}^2}) = H^0(\omega_{X|\mathbb{P}^2}) = H^0(\mathcal{O}_X(7K_X))$. By computing $H^0(\mathcal{O}_X(7K_X))$, according to the computation of case $m = 7$ in Table 3, we get

$$H^0(\mathcal{O}_X(7K_X)) = (x\langle z_1^2, z_1z_2, z_2^2\rangle \mathcal{S}^0V \oplus z_1\mathcal{S}^2V \oplus z_2\mathcal{S}^2V \oplus x\mathcal{S}^3V)/(f_7, f_8),$$

where $V = \langle u, y_1, y_2\rangle$ is the vector space generated by three vectors $u = x^2, y_1, y_2$. After eliminating the two generators z_1^2, z_2^2 via the relations f_7, f_8 like in Proposition 1.17 and taking into account of (1.9) that $\varrho_*\omega_{X|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}$. Moreover the locally free sheaf $\mathcal{E}$ splits into the direct sum of $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$. We get that the module E can be written as the direct sum $E = z_1R(2) \oplus z_2R(2) \oplus xR(3)$. $\square$

Theorem 1.19 — *Let $X \to \mathbb{P}^2$ be a quadruple cover of $\mathbb{P}^2$ which is given by two equations*

$$\begin{aligned} f &= z_1^2 + xz_2a_1(x, y_1, y_2) + b_1(x, y_1, y_2) \\ g &= z_2^2 + xz_1a_2(x, y_1, y_2) + b_2(x, y_1, y_2), \end{aligned}$$

where X is canonically embedded as a complete intersection of bidegree $(6, 6)$ in (the smooth locus of) $\mathbb{P}(1, 2, 2, 3, 3)$. Then:

1. *The bidouble cover $\varrho\colon X \to \mathbb{P}^2$ determines a locally free sheaf of rank 2 and over the polynomial ring $R = \mathbb{C}[x^2, y_1, y_2]$ the module F with respect to sheaf $\mathcal{F} \subseteq S^2\mathcal{E}$ is presented by the following relations:*

$$\begin{aligned} c_1 &= (z_1^2 u + x z_2 u a_1 + x^2 b_1)R(3) \\ c_2 &= (z_2^2 u + x z_1 u a_2 + x^2 b_2)R(3) \\ l &= (z_1^2 b_2 + z_2^2 b_1 - a_2 b_1 x z_1 - a_1 b_2 x z_2)R(1) \end{aligned}$$

and F fits into exact sequence:

$$0 \to F \xrightarrow{\left(\begin{smallmatrix} l & 0 & -c_1 \\ 0 & l & -c_2 \end{smallmatrix}\right)} R(4)^2 \oplus R(6) \xrightarrow{\left(\begin{smallmatrix} c_1 \\ c_2 \\ l \end{smallmatrix}\right)} R(7) \to 0.$$

Proof. By following the description of Casnati and Ekedahl in Section 1.3, we would like to translate these data to the language of the canonical ring in 1.2 as well as the bi-canonical description. We recall that $R = \mathbb{C}[x^2, y_1, y_2]$ and $S = \mathbb{C}[x, y_1, y_2, z_1, z_2]$ are polynomial rings, where x, y_i and z_i have degree 1, 2 and 3, respectively. Let $\varrho\colon X \to \mathbb{P}^2 = \mathsf{Proj}(\mathbb{C}[x^2, y_1, y_2])$ be a quadruple cover, $\mathcal{E}$ be a locally free sheaf of rank 3 over $\mathbb{P}^2$. Then by [CE96] such a embedding $\iota\colon X \hookrightarrow \mathbb{P}^2$ is equivalent to giving a surjective map $\varrho^*\mathcal{E} \to \iota^*\mathcal{O}_{\mathbb{P}(\mathcal{E})} = \omega_{X|\mathbb{P}^2}$ where $\omega_{X|\mathbb{P}^2}$ is relative line bundle and $\omega_{X|\mathbb{P}^2} = \mathcal{O}_X(7K_X)$. Instead of $\varrho^*\mathcal{E} \to \iota^*\mathcal{O}_{\mathbb{P}(\mathcal{E})} = \omega_{X|\mathbb{P}^2}$, we consider the adjoint map $\mathcal{E} \to \varrho_*\omega_{X|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{E}$ and work on the level of graded R-modules.

Writing $E = z_1R(2) \oplus z_2R(2) \oplus xR(3)$ by Proposition 1.18 one computes its second symmetric power to be

$$S^2E = z_1^2R(4) \oplus z_1z_2R(4) \oplus z_2^2R(4) \oplus xz_1R(5) \oplus xz_2R(5) \oplus x^2R(6).$$

We have seen above that $\mathcal{F} = \ker(S^2\mathcal{E} \to \varrho_*\omega_{X|\mathbb{P}^2}^{\otimes 2})$, which for graded modules translates to an exact sequence

$$0 \to F \to S^2E \xrightarrow{\Phi} \Gamma_*(\varrho_*\omega^2_{X|\mathbb{P}^2})$$

To compute F, we write down the R-module $\Gamma(\varrho_*\omega^2_{X|\mathbb{P}^2})$, here we use the description from [Har77, Chap.2, Sect.5]:

$$\begin{aligned} \Gamma_*(\varrho_*\omega^2_{X|\mathbb{P}^2}) &= \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \varrho_*\omega^2_{X|\mathbb{P}^2}(d)) \\ &= \bigoplus_{d\in\mathbb{Z}} H^0(X, (14+2d)K_X) \\ &= \left(\bigoplus_{d\in\mathbb{Z}} H^0(X, 2dK_X)\right)(7) \\ &= R(X, 2K_X)(7) \end{aligned}$$

According to Proposition 1.17, $R(2K_X) \cong \mathbb{C}[u, y_1, y_2, v_1, v_2, w]/(f_1, \ldots, f_6)$. Based on the above descriptions we have the following diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & J \arrow[d] & \\
0 \arrow[r] & F \arrow[r] \arrow[d, equal] & S^2E \arrow[r, hook] \arrow[d, equal] & \mathbb{C}[u = x^2, y_1, y_2][v_1, v_2, w, z_1^2, z_2^2](7) \arrow[d] & \\
0 \arrow[r] & F \arrow[r, hook, "\varphi"] & S^2E \arrow[r] & \mathbb{C}[u = x^2, y_1, y_2][v_1, v_2, w, z_1^2, z_2^2](7)/J \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

It is easy to see that $F = J \cap S^2E$, thus, F is a submodule of S^2E generated by

$$\begin{aligned}
(z_1^2u + xz_2ua_2 + x^2b_1)R(3) &= c_1R(3), \\
(z_2^2u + xz_1ua_1 + x^2b_2)R(3) &= c_2R(3), \\
(z_1^2b_2 - z_2^2b_1 - a_2b_1xz_1 + a_1b_2xz_2)R(1) &= lR(1).
\end{aligned}$$

The presentation of F can be understood from the minimal exact sequence of F

$$0 \to F \to R(4)^{\oplus 2} \oplus R(6) \to R(7) \to 0$$

by taking dual of this exact sequence and taking into account that $\mathcal{F} \cong \check{\mathcal{F}}(7)$, then

$$0 \to R \xrightarrow{(\,-u\ b_2\ b_1\,)^T} R(1) \oplus R(3) \oplus R(3) \xrightarrow{\begin{pmatrix} l \\ c_1 \\ c_2 \end{pmatrix}^T} F \to 0$$

Thus $F = \langle c_1, c_2, l \rangle \subseteq S^2E$. $\square$

Remark 1.20 — Locally, X can be seen as a complete intersection of two relative conics. Cover $\mathbb{P}^2$ by the open subsets $D_+(u), D_+(y_ib_j)$ where $D_+(y_1) = D_+(y_1b_1) \cup D_+(y_1b_2); D_+(y_2) = D_+(y_2b_1) \cup D_+(y_2b_2)$.

Over $D_+(u) = \mathbb{A}^2$ we have

$$\begin{aligned}
\mathcal{E}_{|_{D_+(u)}} &= \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)_{|_{D_+(u)}} \\
&= ((S(\tilde{2})_u))_0 \oplus ((S(\tilde{2})_u))_0 \oplus ((S(\tilde{3})_u))_0 \\
\mathcal{E}_{|_{D_+(u)}}(D_+(u)) &= u^2(S_u)_0 \oplus u^2(S_u)_0 \oplus u^3(S_u)_0
\end{aligned}$$

Thus we can write the equations defining the module F on $D_+(u)$ from the presentation of F, for example

$$c_1 = (z_1^2u + xz_2ua_1 + x^2b_1)R(3)$$

Note that $\deg z_i = -2$ in $S(2)$ and $\deg x = -3$ in $S(3)$. The polynomial can be writen as

$$(z_1u^2)^2 + (xu^3)(z_2u^2)a_1\left(\frac{y_1}{u}, \frac{y_2}{u}\right) + (xu^3)^2b_1\left(\frac{y_1}{u}, \frac{y_2}{u}\right) = 0$$

By setting $Z_1 = z_1 u^2; X = xu^3; Z_2 = z_2 u^2$. Then $\deg Z_i = \deg X = 0$. The equation is now

$$Z_1^2 + XZ_2 A_1 + X^2 B_1 = 0$$

thus we get the other 2 equations from the presentation of F:

$$\begin{aligned} Z_2^2 + XZ_1 A_2 + X^2 B_2 &= 0, \\ Z_1^2 B_1 - Z_2^2 B_2 - A_2 B_1 X Z_1 - A_1 B_2 X Z_2 &= 0. \end{aligned}$$

The third equation vanishes automatically, so on $D_+(u)$ the local equations presenting F are the two following:

$$\begin{aligned} Z_1^2 + XZ_2 A_1(Y_1, Y_2) + X^2 B_1(Y_1, Y_2) &= 0, \\ Z_2^2 + XZ_1 A_2(Y_1, Y_2) + X^2 B_2(Y_1, Y_2) &= 0. \end{aligned}$$

Similary we can compute for the other open subset $D_+(y_i b_j)$. Note that the module $\mathcal{F} \subseteq S^2\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(4)^{\otimes 3} \oplus \mathcal{O}_{\mathbb{P}^2}(5)^{\otimes 2} \oplus \mathcal{O}_{\mathbb{P}^2}(6)$ and

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^2}(4)_{|_{D_+(y_i b_j)}}(D_+(y_i b_j)) &= (\tilde{R}(4)_{y_i b_j})_0 = y_i b_j((R_{y_i})_{b_j})_0, \\ \mathcal{O}_{\mathbb{P}^2}(5)_{|_{D_+(y_i b_j)}}(D_+(y_i b_j)) &= (\tilde{R}(5)_{y_i b_j})_0 = y_i^2 b_j((R_{y_i})_{b_j})_0, \\ \mathcal{O}_{\mathbb{P}^2}(4)_{|_{D_+(y_i b_j)}}(D_+(y_i b_j)) &= (\tilde{R}(6)_{y_i b_j})_0 = b_j^2((R_{y_i})_{b_j})_0. \end{aligned}$$

Thus, for example in $D_+(y_1 b_1)$, we get two following equations:

$$\begin{aligned} Z_1^2 U + Z_2 XUA_1 + X^2 &= 0, \\ Z_2^2 U + Z_1 XUA_2 + X^2 B_2 &= 0, \\ Z_1^2 B_2 - Z_2^2 - A_2 X Z_1 - A_1 B_2 X Z_2 &= 0, \end{aligned}$$

of which the second one is redundant.

1.6. Building data of the bi-double covers

It is shown in 1.8 that X is a bi-double cover if and only if, up to a coordinate change, the terms a_1 and a_2 in the equations of Theorem 1.6 vanish. We recall from [FPR17, AP12] that the bi-double cover $\varrho\colon X \to \mathbb{P}^2$ is uniquely determined (up to isomorphism of covers) by effective divisors D_i of $\mathbb{P}^2$ of degree $d_i, i = 0, 1, 2$, such that:

1. $d_i \equiv d_j \mod 2$ for every i, j.
2. the so-called Hurwitz divisor $\Delta := \frac{1}{2}(D_0 + D_1 + D_2)$ has no component of the multiplicity > 1.

The divisors D_0, D_1 and D_2 are called the building or branch data of ϱ. Setting $a_i = \frac{d_j + d_k}{2}$ where i, j, k is a permutation of $0, 1, 2$, one has

$$\varrho_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-a_0) \oplus \mathcal{O}_{\mathbb{P}^2}(-a_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-a_2)$$

The conditions $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$ imply that $(d_0, d_1, d_2) = (1, 3, 3)$, that is, $(a_0, a_1, a_2) = (3, 2, 2)$ which confirms $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$ in the notation of Proposition 1.18.

Remark 1.21 — A bi-double cover with branch data D_0, D_1, D_2 can be seen as an iterated double cover as follows. First one takes the double cover $f\colon Y \to \mathbb{P}^2$ branched on D_0+D_1: If D_0 and D_1 intersect transversally, then Y is a singular Del Pezzo surface of degree 2 that has ordinary double points over three intersection points of D_0 and D_1. The cover $g\colon X \to Y$ is obtained by taking the double cover branched over the singular points and over the divisor $B := f^*D_2 \in |-3K_Y|$.

We now give some examples of the bi-double covers by describing the branch data D_i. We take the coordinates (u, y_1, y_2) in $\mathbb{P}^2$ so that the first branch divisor is the line $D_0 = \{u = 0\}$ and only the cubics D_1 and D_2 are specified.

Elliptic singularities of degree 1

This case happens when $D_1 + D_2 + D_3$ has an ordinary quadruple point at P, such that three of the local components are in the same D_i.

For example, let D_1 be a union of three general lines through $P \in D_0$ and let D_2 be a general cubic like in Figure 1. We consider the bi-double cover branched over $D_0 + D_1 + D_2$: we take a double cover branched over $D_0 + D_1$ and another double cover branching over D_2. By blowing up at P, the branched data now become $\tilde{D}_0 + E$, $\tilde{D}_1 + 3E$ and $\tilde{D}_3$. It is easy to see that the bi-double cover is singular along a divisor E, and thus it is not normal. We would like to normalise it by changing the branched data following [FPR17] Proposition 5.1 and Remark 5.2. Indeed, after blowing up at the point P, we first reduce $\tilde{D}_i$ modulo 2 and then remove all the irreducible components that are common to all $\tilde{D}_i$. We get new branch data $\tilde{D}_0, \tilde{D}_1$ and $\tilde{D}_3 + E$. On the first double cover $\sigma_1\colon X_1 \to \mathbb{P}^2$ branched over $\tilde{D}_0 + \tilde{D}_1$, one gets an elliptic curve $E_1 = \sigma_1^*\tilde{D}_3$ with $E_1^2 = (\sigma_1^*\tilde{D}_3)^2 = (\tilde{D}_3)^2.\deg(\sigma) = -2$. Similarly, for the double cover $\sigma_2\colon X_1 \to X$ we obtain an elliptic curve $E = \frac{1}{2}\sigma_2^* E_1$ since E_1 is in the branch and $E^2 = (\sigma_2^*E_1)^2 = E_1^2 \deg(\sigma_2) = -1$. In this way, we show that there is only one elliptic singularity of degree 1.

Elliptic singuarities of degree 2

An elliptic singularity of degree 2 can be seen as an iterated bi-double cover of $\mathbb{P}^2$ by taking D_1 general and choosing B with a quadruple point a smooth point Q of Y such that the infinitely near points are at most double. We obtain an example with an elliptic Gorenstein singularity of degree 2, see Figure 2.

In this case, we need two blow-ups, in which the branch data are changed as in the case elliptic singularity of degree 1. After two blow-ups and changing the branch data, we can consider it as a iterated bi-double cover.

Indeed, first we take a double cover σ branched over $D_1'' + D_2''$, then the pullback of the E_i lies on the branch. Let $F_i := \sigma^*E_i$, we have $F_i^2 = 2E_i^2$. $E' := F_1 + F_2$ is an elliptic curve with degree

$$\begin{aligned}
E'^2 = (F_1 + F_2)^2 &= F_1^2 + F_2^2 + 2F_1F_2 \\
&= (\sigma^*E_1)^2 + (\sigma^*E_2)^2 + 2(\sigma^*E_1)(\sigma^*E_2) \\
&= 2E_1^2 + 2E_2^2 + 2E_1E_2 \\
&= -4 - 2 + (-2).2 = -2
\end{aligned}$$

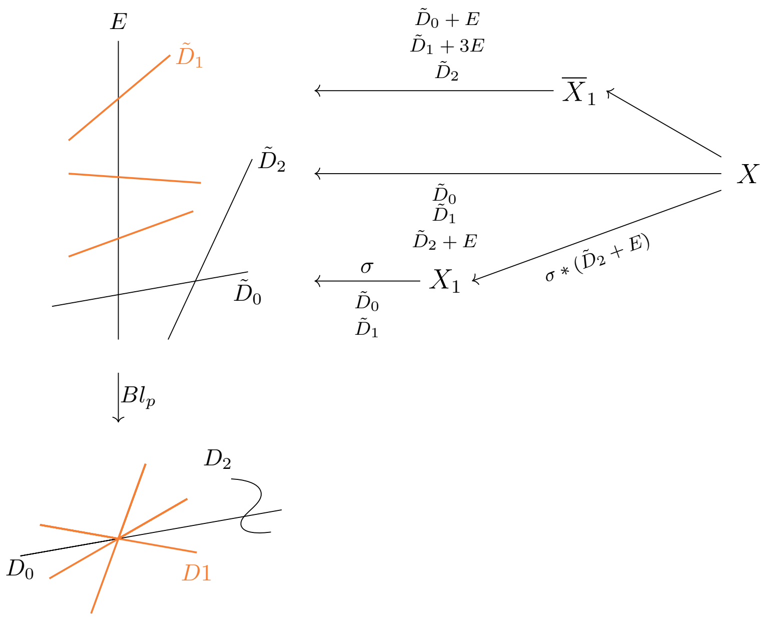


Figure 1: Elliptic singularity of degree 1.

A second double cover is branched over the singularities of the first cover and $\sigma^*(D_0'' + E_1')$. We obtain an elliptic curve $E = \frac{1}{2}\sigma'^* E'$ since E' is on the branch and $E^2 = -2$. In this case, we showed that there is a cursp singularity of degree 2.

A divisor B exists and can be seen by taking $Y = \{y^2 - x_0(x_0^3 + x_1^3 + x_2^3 + 2x_0x_2^2)\}$ and $B \subset Y$ given by $x_1(y + x_0^2 + x_2^2)$.

Elliptic singularities of degree 4

If we take D_1 and D_2 such that both of them have an ordinary double point at P and D has an ordinary quadruple point at P like in Figure 3, the resulting singularity is an elliptic singularity of degree 4. An explanation of this type of singularity is provided in the example of Section 2.1.

2. Local analysis of quadruple covers

In this section we are going to study local properties of the quadruple cover. It was proven in [CE96] that if $X \to \mathbb{P}^2$ is a quadruple cover, then X is embedded in $\mathbb{P}^2$-bundle and each fibre consists of four points in $\mathbb{A}^2$ which are intersections of two conics.

Lemma 2.1 — *Let X_0 be a fiber of the Gorenstein quadruple cover $X \to \mathbb{P}^2$. There are five possibilities of its fibres, namely type I, II_a, II_b, III, IV_a and IV_b as described in Table 4.*

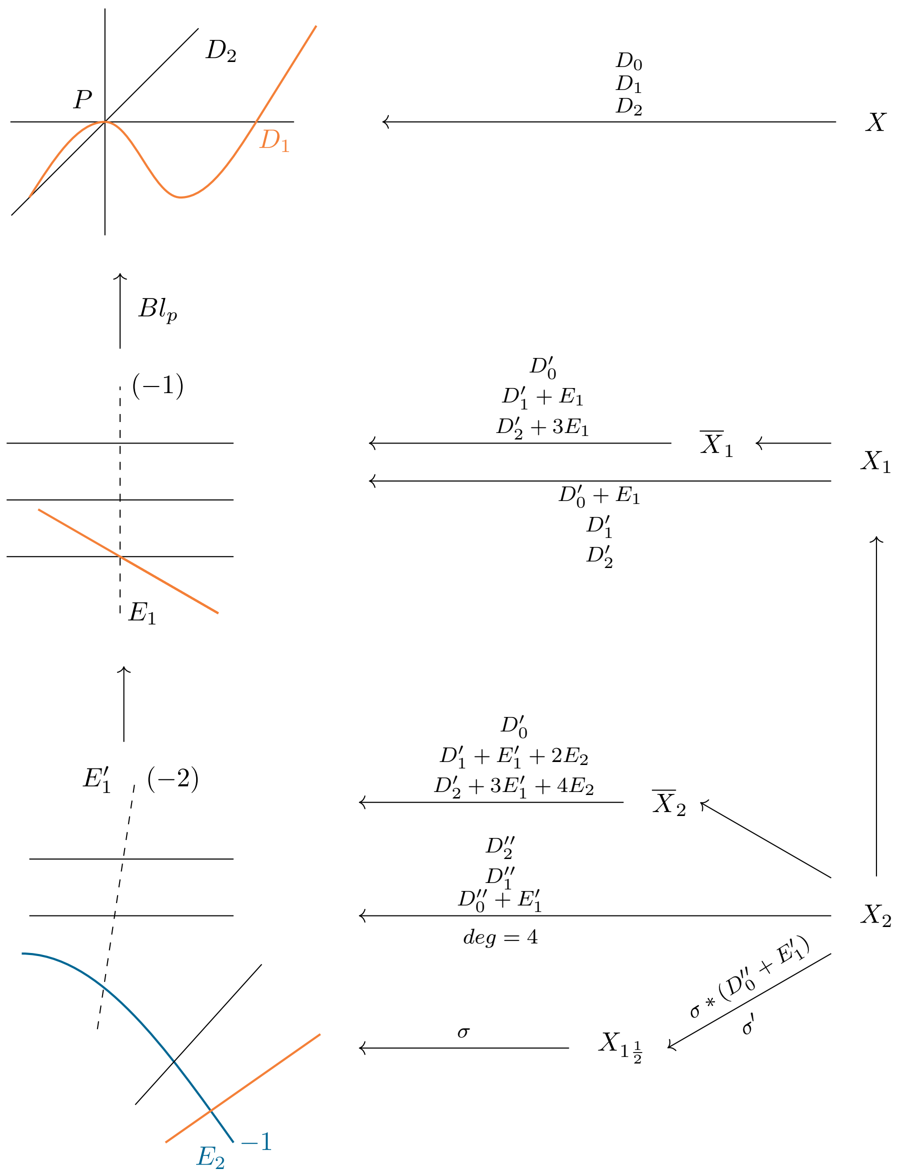


Figure 2: Elliptic singularity of degree 2

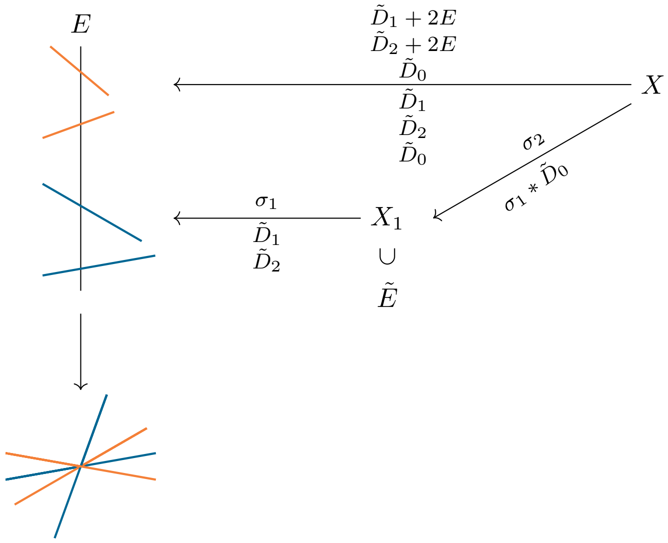


Figure 3: Elliptic singularity of degree 4.

Proof. It follows from the classification of the commutative, associate algebra of rank 4 over $\mathbb{C}$ in [HM99, Table 6.1]. ☐

Remark 2.2 — Let $\varrho \colon X \to \mathbb{P}^2$ be a quadruple cover and let its local equations be described as in Theorem 1.19. Then there exists an étale, formal or analytic open neighborhood $U \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2$ of 0 such that $X_U = Z(f, g)$. The fibre $X_0 = (f_0, g_0)$ is smooth if and only if X_0 is type I.

Proof. It follows from the equations in Theorem 1.19 and the Jacobian criterion for singularities. ☐

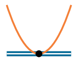
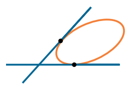
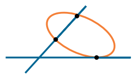
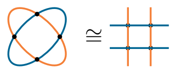
2.1. Local description near a type IV_a fibre

Let $X \to \mathbb{P}^2$ be a quadruple cover. Let $U \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2$ be a formal or analytic neighbor of 0 such that the fibre X_0 is of type IV_a and its local equation as in Table 4. Since X_0 is a complete intersection of two conics, we can apply the theory of deformation of a complete intersection which is described in [AST76]. We would like to apply to the embedded deformation of X_0 via description in [AST76], the following diagram comes from the definition of the deformation

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ X_0 \arrow[r, "\mu"] & \mathrm{Def}(\bullet) \cong \mathbb{A}^8 \end{tikzcd}$$

Let $X_0 \hookrightarrow \mathbb{A}^2$ is a complete intersection with $\dim X_0 = 0$ and let $I = (z_1^2, z_2^2)$ be the ideal of X_0. Moreover, X_0 is a length 4 algebra over $\mathbb{C}$ which is spanned by

Table 4: Gorenstein $\mathbb{C}$ - algebra of length 4

Type	Symbol	Figures	Example for equations
IV_a	$\bullet^4$		$\mathbb{C}[z_1, z_2]/(z_1^2, z_2^2)$
IV_b	$\bullet^4$		$\mathbb{C}[z_1, z_2]/(z_1^2, z_2^2 + z_1)$
III	$\bullet^3 \quad \bullet^1$		$\mathbb{C}[x, z_2]/(xz_2 - 1, (z_2 - x)(z_2 - x + a_2b_2))$
II_b	$\bullet^2 \quad \bullet^2$		$\mathbb{C}[z_1, z_2]/(z_1^2, z_2^2 + z_1a_2 + b_2)$
II_a	$\bullet^2 \bullet^1 \bullet^1$		$\mathbb{C}[z_1, z_2]/(z_1^2 + z_2a_1, z_2^2 + z_1a_2)$
I	$\bullet^1\bullet^1\bullet^1\bullet^1$		$\mathbb{C}[z_1, z_2]/(z_1^2 + b_1, z_2^2 + z_1a_2)$

$1, z_1, z_2$ and z_1z_2. We would consider the embedded deformation parameterized by these generators in $\mathbb{A}^8$. Then the embedded deformation of X_0 in $\mathbb{A}^8$ has equations :

$$\begin{aligned} z_1^2 + a_1 + b_1z_1 + c_1z_2 + d_1z_1z_2 = 0, \\ z_2^2 + a_2 + b_2z_1 + c_2z_2 + d_2z_1z_2 = 0, \end{aligned} \tag{2.3}$$

over an open subset $U = \mathbb{A}^2$ with $a_1, ..., d_2 \in (u, v)$. Here we assume that the central fibre is of type IV_a.

Lemma 2.4 — *Let denote $0 = (0, 0)$ in U. Then*

1. *X is smooth over near $\varrho^{-1}(0) = X_0$ if and only if*

$$\Delta = \det \begin{pmatrix} \partial_u a_1 & \partial_v a_1 \\ \partial_u a_2 & \partial_v a_2 \end{pmatrix} \neq 0$$

at $(0, 0)$.

2. *X has an isolated singularity at 0 if and only if $\Delta(0, 0) = 0$ and*

$$T := \mathbb{C}[z_1, z_2, u, v]/(z_1^2, z_2^2, \Delta) \cong \mathbb{C}$$

Proof. We rewrite the equations of X as

$$\begin{aligned} f_1 = z_1^2 + r_1(u, v, z_1, z_2), \\ f_2 = z_2^2 + r_2(u, v, z_1, z_2), \end{aligned} \tag{2.5}$$

where $r_1(u, v, z_1, z_2) = a_1 + b_1z_1 + c_1z_2 + d_1z_1z_2$ and $r_2(u, v, z_1, z_2) = a_2 + b_2z_1 + c_2z_2 + d_2z_1z_2$. The singular locus of X is given by $Z(f, g, \mathsf{rk}\,\Delta_1 \leq 2)$, where

$$\Delta_1 = \begin{pmatrix} 2z_1 + b_1 + d_1z_2 & c_1 + d_1z_1 & \partial_u r_1 & \partial_v r_1, \\ b_2 + d_2z_2 & 2z_2 + c_2 + d_2z_1 & \partial_u r_2 & \partial_v r_2, \end{pmatrix}.$$

Since $a_1, .., d_2 \in (u, v)$, we have the following at $(0, 0)$

$$\Delta_1(0,0) = \begin{pmatrix} 0 & 0 & \partial_u a_1 & \partial_v a_1 \\ 0 & 0 & \partial_u a_2 & \partial_v a_2 \end{pmatrix}$$

Therefore, X is smooth near 0 if and only if

$$\Delta = \det \begin{pmatrix} \partial_u a_1 & \partial_v a_1 \\ \partial_u a_2 & \partial_v a_2 \end{pmatrix} \neq 0$$

at $(0, 0)$. If $\Delta(0, 0) = 0$, then X is singular near (0,0). We can apply the deformations of complete intersection as in [AST76, Section 4] that X has isolated singularity at 0 if $T := \mathbb{C}[z_1, z_2, u, v]/(z_1^2, z_2^2, \Delta) \cong \mathbb{C}$. $\square$

We now assume that X is singular over 0. It is of interest to know whether the first blow up at $0 \in \mathbb{A}^2_{u,v}$ of X is smooth as well as Gorenstein. For this purpose we let $k_i = \min\{\mathsf{mult}_0(a_i), ..., \mathsf{mult}_0(d_i)\}$. With the previous assumption that $\Delta \neq 0$ we have $k_i \geq 1$ for every i.

Consider the following diagram, where $\tilde{U} \to U$ is blow up of U at the origin. Let $X' = \tilde{U} \times_U X$ be the fibre product, X'' be the partial normalisation constructed below and $\tilde{X}$ be the normalisation:

$$\begin{tikzcd}
\tilde{X} \arrow[dr] \arrow[drr] \arrow[ddr] & & & \\
& X'' \arrow[r] \arrow[d] & X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\
& \tilde{U} \arrow[r, equal] & \tilde{U} \arrow[r, "Bl"] & U
\end{tikzcd}$$

For the computation we follow the Equation (2.3) of X in $\mathfrak{X}$. By considering the blow up of U at $0 \in \mathbb{A}^2$ and looking at a part of its strict transform, we obtain the equations of $X' = X \times_U \tilde{U} \subset \mathbb{A}^2_{u,\overline{v}} \times \mathbb{A}^2_{z_1,z_2}$

$$\begin{aligned}
z_1^2 + a_1(u, u\overline{v}) + b_1(u, u\overline{v})z_1 + c_1(u, u\overline{v})z_2 + d_1(u, u\overline{v})z_1 z_2 = 0 \\
z_2^2 + a_2(u, u\overline{v}) + b_2(u, u\overline{v})z_1 + c_2(u, u\overline{v})z_2 + d_2(u, u\overline{v})z_1 z_2 = 0
\end{aligned}$$

It is easy to see from the equations that X' is singular along line $(z_1 = z_2 = u = 0)$, therefore X' is not normal. To normalise X' we first write the two equations in the forms $z_1^2 + u^{k_1}\tilde{r}_1, z_2^2 + u^{k_2}\tilde{r}_2$. We would like to normalise X' by letting $\tilde{z_1} = \frac{z_1}{u^i}, \tilde{y} = \frac{z_2}{u^j}$, where $i := \lfloor \frac{k_1}{2} \rfloor$ and $j := \lfloor \frac{k_2}{2} \rfloor$, by this way we get equations for a partial normalisation X''

$$\begin{aligned}
\tilde{z_1}^2 + u^{\varepsilon_1}(\tilde{a}_1 + \tilde{b}_1\tilde{z}_1 u^i + \tilde{c}_1\tilde{z}_2 u^j + \tilde{d}_1\tilde{z}_1\tilde{z}_2 u^{i+j}) = 0 \\
\tilde{z_2}^2 + u^{\varepsilon_2}(\tilde{a}_2 + \tilde{b}_2\tilde{z}_1 u^i + \tilde{c}_2\tilde{z}_2 u^j + \tilde{d}_2\tilde{z}_1\tilde{z}_2 u^{i+j}) = 0
\end{aligned} \tag{2.6}$$

If k_1 and k_2 are both odd numbers, then X'' is not normal and we need to normalise in further steps, for example in the diagram above we get $\tilde{X}$ as the second step of the normalisation. Otherwise, X'' is normal as the first step of normalisation. We have the following as the first results of the local descriptions near type IV_a fibres.

2.2. First step partial nomalisation

The first observation is that if k_1 and k_2 are not both odd numbers, then the local fiber is normal after a first natural step towards normalisation. In this case we have $\varepsilon_1 = \varepsilon_2 = 0$ or $\varepsilon_1 = 0, \varepsilon_2 \neq 0$ in the equation (2.6). We consider first the case of $\varepsilon_1 = \varepsilon_2 = 0$.

Proposition 2.7 — *Assume that $k_1, k_2 \equiv 0 \mod 2$, then we have the following:*

1. *X'' is smooth if and only if $\tilde{a}_1\tilde{a}_2$ has no multiple zeros,*
2. *If X'' is singular, then it can be in the following cases:*
   a) *If $\tilde{a}_1$ and $\tilde{a}_2$ have common zeros but Δ does not have full rank, then X'' has type IV_a at the center fibre.*
   b) *If $\tilde{a}_1$ and $\tilde{a}_2$ have no common zero, then the center fibre of $\tilde{X}$ is of type II_a.*

Proof. On the first open subset X''_1 has equations

$$\tilde{z_1}^2 + \tilde{a}_1 + u\tilde{b}_1\tilde{z}_1 + u\tilde{c}_1\tilde{z}_2 + u^2\tilde{d}_1\tilde{z}_1\tilde{z}_2 = 0,$$
$$\tilde{z_2}^2 + \tilde{a}_2 + u\tilde{b}_2\tilde{z}_1 + u\tilde{c}_2\tilde{z}_2 + u^2\tilde{d}_2\tilde{z}_1\tilde{z}_2 = 0,$$

here we assume that $\tilde{a}_1$ and $\tilde{a}_2$ are not divisible by u, the Jacobian matrix of X''_1 along $u = 0$ is

$$\Delta = \begin{pmatrix} 2\tilde{z}_1 & 0 & \frac{\partial \tilde{a}_1}{\partial u} + \tilde{c}_1\tilde{z}_2 & \frac{\partial \tilde{a}_1}{\partial v} \\ 0 & 2\tilde{z}_2 & \frac{\partial \tilde{a}_2}{\partial u} + \tilde{b}_1\tilde{z}_1 & \frac{\partial \tilde{a}_2}{\partial v} \end{pmatrix}$$

Note that the singular locus of X''_1 is $X''_{1,\text{sing}} = Z(\tilde{z}_1^2 + \tilde{a}_1, \tilde{z}_2^2 + \tilde{a}_2, \Delta)$, and one of the conditions that Δ does not have full rank is $4\tilde{z}_1\tilde{z}_2 = 0$. We have the following situations:

- If $\tilde{z}_1 = 0$. Then the singular locus of X''_1 is vanishing locus of following equations

$$\begin{aligned} \tilde{a}_1 &= 0 \\ \tilde{z}_2^2 + \tilde{a}_2 &= 0 \\ \tilde{z}_2(\frac{\partial \tilde{a}_1}{\partial u} + \tilde{c}_1\tilde{z}_2) &= 0 \\ \tilde{z}_2\frac{\partial \tilde{a}_1}{\partial v} &= 0 \\ (\frac{\partial \tilde{a}_1}{\partial u} + \tilde{c}_1\tilde{z}_2)\frac{\partial \tilde{a}_2}{\partial v} - \frac{\partial \tilde{a}_2}{\partial u}\frac{\partial \tilde{a}_1}{\partial v} &= 0 \end{aligned}$$

  If $\tilde{z}_1 = \tilde{z}_2 = 0$, we get $\tilde{a}_1(0, v) = \tilde{a}_2(0, v) = 0$ and

$$\det \begin{pmatrix} \partial_u\tilde{a}_1 & \partial_v\tilde{a}_1 \\ \partial_u\tilde{a}_2 & \partial_v\tilde{a}_2 \end{pmatrix} \neq 0$$

  On another open subset we get $\tilde{a}_1(u, 0) = \tilde{a}_2(u, 0) = 0$ and

$$\det \begin{pmatrix} \partial_u\tilde{a}_1 & \partial_v\tilde{a}_1 \\ \partial_u\tilde{a}_2 & \partial_v\tilde{a}_2 \end{pmatrix} \neq 0.$$

  Thus X'' is smooth if $\tilde{a}_1\tilde{a}_2$ has no multiply zeros at $(0, 0)$

- If $\tilde{z}_2 \neq 0$, then $\tilde{a}_2 \neq 0$. Thus $\tilde{z}_2^2 + \tilde{a}_2 = 0$ and $\tilde{a}_1 = 0$. This implies $\frac{\partial(\tilde{a}_2)}{\partial v} = 0$. Similar for an other open subset we get X is smooth if $\tilde{a}_2$ has no multiply zeros at $(0,0)$.

On other chart of the blow-up, namely when $v \neq 0$, we get a similar description of X'' on this open subset. Thus the conditions for X'' be smooth and also singular coincide with the condition on the path when $u \neq 0$.

□

Proposition 2.8 — *If* $(k_1, k_2) \equiv (0,1) \mod 2$ *then* X'' *is singular over the zero set of* $\tilde{a}_2$.

Proof. If $(k_1, k_2) \equiv (0,1) \mod 2$, then over a part U_1'', X_1'' has the following equations:

$$\tilde{z}_1^2 + \tilde{a}_1 + u\tilde{b}_1\tilde{z}_1 + \tilde{c}_1\tilde{z}_2 + u\tilde{d}_1\tilde{z}_1\tilde{z}_2 = 0,$$
$$\tilde{z}_2^2 + u(\tilde{a}_2 + u\tilde{b}_2\tilde{z}_1 + \tilde{c}_2\tilde{z}_2 + u\tilde{d}_2\tilde{z}_1\tilde{z}_2) = 0.$$

Here we assume that $\tilde{a}_1$ and $\tilde{a}_2$ are not divisible by u. $X''_{sing} \cap \{u = 0\}$ is given by the following:

$$\tilde{z}_1^2 + \tilde{a}_1 + \tilde{c}_1\tilde{z}_2 = 0,$$
$$\tilde{z}_2^2 = 0,$$

$$\mathsf{rk}\begin{pmatrix} 2\tilde{z}_1 & \tilde{c}_1 & \frac{\partial \tilde{a}_1}{\partial u} + \tilde{b}_1\tilde{z}_1 & \frac{\partial \tilde{a}_1}{\partial v} \\ 0 & 0 & \tilde{a}_2 & 0 \end{pmatrix} < 2,$$

which is equivalent to $z_1^2 + \tilde{a}_1 = \tilde{z}_2 = \tilde{a}_2\tilde{z}_1 = \tilde{a}_2\frac{\partial \tilde{a}_1}{\partial v} = \tilde{a}_2\tilde{c}_1 = 0$. We have $Z(\tilde{a}_1, \tilde{a}_2) \subset Z(\tilde{a}_2) \subset X_{\text{Sing}} \subset Z(\tilde{a}_1\tilde{a}_2)$. If $\tilde{a}_2(0,v)$ is non zero every where,i.e., $\tilde{a}_2(0,v) = \text{constant} \neq 0$, but then by considering the other path we get $c = 0$. Thus this case does not happen. Assume that $P \in Z(\tilde{a}_2)$, then P is a singular point of X''. □

2.2.1. Log canonical singularities

Now we assume that X is lc. By computing the discrepancy along the exceptional curve appearing in X'' we will show that this restricts severely the possible values of (k_1, k_2). We use the equation 2.6 where where $i := \lfloor \frac{k_1}{2} \rfloor$ and $j := \lfloor \frac{k_2}{2} \rfloor$.

We write $K_{X''} = \pi^* K_X + \sum a_i E_i$. For the computation of $K_{X''}$ we note that $K_X = div(\varrho^* du \wedge dv)$. According to [LE06] Corollary 4.14

$$\omega_X = \frac{1}{\Delta} du \wedge dv$$

where

$$\Delta = \det\begin{pmatrix} 2z_1 + b_1 + d_1 z_2 & d_1 z_1 + c_1 \\ d_2 z_2 + b_2 & 2z_2 + c_2 + d_2 z_1 \end{pmatrix}$$

and the pullback of the canonical divisor K_X along π can be computed as the following

$$\begin{aligned}\pi^* K_X &= div\left(\pi^*\left(\frac{1}{\Delta} du \wedge dv\right)\right)\\ &= div\left(\left(\frac{1}{\pi^*\Delta} du \wedge d(u\overline{v})\right)\right)\\ &= div\left(\left(\frac{1}{\pi^*\Delta} u.du \wedge d(\overline{v})\right)\right)\\ &= div\left(\left(\frac{u^{1-i-j}}{\tilde{\Delta}} du \wedge d(\overline{v})\right)\right)\end{aligned}$$

Thus $K_{X''} = \pi^* K_X - (i+j-1)E = \pi^* K_X - (\lfloor \frac{k_1}{2} \rfloor + \lfloor \frac{k_2}{2} \rfloor - 1)E$. Note that X is lc if and only if the discrepancy $\lfloor \frac{k_1}{2} \rfloor + \lfloor \frac{k_2}{2} \rfloor - 1 \leq 1, \quad k_1, k_2 \geq 1$, thus there are some possibilities as in following list, where the case of k_1 and k_2 being odd is excluded by our assumptions.

- $(k_1, k_2) = (2, 2)$,
- $(k_1, k_2) = (2, 1), (2, 3), (4, 1)$

We look at these cases more closely under the assumption that X'' is smooth.

Proposition 2.9 — *Assume X is slc, X'' smooth and the central fibre is defined by $Z(z_1^2, z_2^2)$. There is 3 possible cases:*

1. *X has ADE singularities if $(k_1, k_2) = (2, 1), (2, 3), (4, 1)$,*
2. *X has elliptic singularities of degree 4 if $(k_1, k_2) = (2, 2)$.*

Proof. Asume that X is slc and X_1'' is smooth after first blow up, then $(k_1, k_2) = \{(2, 1), (4, 1), (2, 3), (2, 2)\}$. We will look at the exceptional divisor in X_1''.

1. If $(k_1, k_2) = (2, 2)$, then the exceptional curve is quadruple cover of $\mathbb{P}^1$ which is defined as the following equations:

   $$\begin{aligned}z_1^2 + a_1 + ub_1 z_1 + uc_1 z_2 + u^2 d_1 z_1 z_2 = 0\\ z_2^2 + a_2 + ub_2 z_1 + uc_2 z_2 + u^2 d_2 z_1 z_2 = 0\end{aligned}$$

   We look at $\tilde{X}_1$ over the exceptional divisor $E = \mathbb{P}^1 = [(0, 0), (\overline{u}, \overline{v})]$. E now has equations:

   $$\begin{aligned}z_1^2 + q_1(\overline{u}, \overline{v}) = 0\\ z_2^2 + q_2(\overline{u}, \overline{v}) = 0\end{aligned}$$

   We would like to compute the genus of E and its self intersection E^2. For $p_a(E)$ we have $p_a(E) = 1 - \chi(\pi_* \mathcal{O}_E) = 1 - \chi(\mathcal{O}_{\mathbb{P}^1} \oplus L_1 \oplus L_2 \oplus L_3) = 1$ ($a_1 a_2$ has no multiple roots.) Now we proof that $E^2 = -4$, it can be illustrated by Figure 3. Similarly, we get the following results:

2. If $(k_1, k_2) = (2, 1), (2, 3), (4, 1)$, then E is not reduced and E_{red} has an ADE singularty.

$\square$

To verify these results, we compute on the examples in Section 1.6 by giving explicit equations for the D_i.

Examples

One example of Type IV where $(k_1, k_2) = (2, 2)$ is the case in Example 1.6, where D_1 and D_2 have an ordinary double point at P such that D has an ordinary quadruple point at P. As an example, we can choose the local equation $D_1 = (y_1 + 2y_2)y_2(u - y_1), D_2 = (y_1^2 - y_2^2)(y_2 - 2u)$. Then X is a bi-double cover of $\mathbb{P}^2$ whose equations are:

$$\begin{aligned} z_1^2 u + x^2(y_1 + 2y_2)y_2(u - y_1) &= 0 \\ z_2^2 u + x^2(y_1^2 - y_2^2)(y_2 - 2u) &= 0 \\ z_1^2(y_1 + 2y_2)y_2(u - y_1) + z_2^2(y_1^2 - y_2^2)(y_2 - 2u) &= 0 \end{aligned}$$

Locally, over $0 = (0, 0) \in \mathbb{A}^2_{y_1,y_2}$, in $\mathbb{A}^2_{z_1,z_2} \times \mathbb{A}^2_{y_1,y_2}$, X is singular and has equation (after localization at the maximal ideal $\mathfrak{m} = (y_1, y_2)$):

$$\begin{aligned} z_1^2 + (y_1 + 2y_2)y_2 &= 0 \\ z_2^2 + (y_1^2 - y_2^2) &= 0 \end{aligned}$$

The blow up of $\mathbb{A}^2_{y_1,y_2}$ at the origin is

$$Bl_0(\mathbb{A}^2) = \{(y_1, y_2)[Y_1 : Y_2] \quad \text{so that} \quad y_1Y_2 = y_2Y_1\}$$

Over an open subset:

$$U_{Y_1} = \{(y_1, y_2)[1 : Y_2] \quad \text{so that} \quad y_1Y_2 = y_2\}$$

$\tilde{X}$ has equations

$$\begin{aligned} z_1^2 + y_1^2(1 - 2Y_2)Y_2 &= 0 \\ z_2^2 + y_1^2(1 - Y_2^2) &= 0 \end{aligned}$$

X_0 is not normal along the exceptional divisor E and its normalisation is of the form ($Z_i := z_i/y_1$):

$$\begin{aligned} Z_1^2 + (1 - 2Y_2)Y_2 &= 0 \\ Z_2^2 + 1 - Y_2^2 &= 0 \end{aligned}$$

we have $K_E = \pi^* K_{\mathbb{P}^1} + \frac{1}{2}(D_1 + D_2) = 0$ thus $g(E) = 1$ an elliptic curve. The Normal bundle over E is $\oplus \mathcal{O}_E(-2)^{\oplus 2}$ and thus E is an elliptic curve of degree 4.

Let D be discriminant locus of X_0, then $D = V((y_1 + 2y_2)y_2(y_1^2 - y_2^2))$ union of four lines through the origin. Blow up $\mathbb{A}^2$ at the origin we see that D becomes smooth after the first blow up and its bi-double cover branched over four points which is also an elliptic curve.

Proposition 2.10 — *A Gorenstein quadruple cover of $\mathbb{A}^2_{y_1,y_2}$ which has elliptic singularity of degree 4 over $(0, 0) \in \mathbb{A}^2$ if locally the multiplicity of $\tilde{b}_i$ at $(0, 0) \in \mathbb{A}^2$ are 2 and the $\tilde{a}_i$ has the multiplicity at least 1 over $(0, 0) \in \mathbb{A}^2$.*

Proof. The proof follows directly from Proposition 2.9 $\square$

2.3. Second step normalisation and non-Gorenstein covers

In this subsection we will consider the case when $\varepsilon_1 = \varepsilon_2 = 1$ in the equation 2.6, then X'' is singular along line $\{u = 0\}$ and thus X'' is not normal with non normal locus given by $\{\tilde{z}_1 = \tilde{z}_2 = u = 0\}$. A second partial normalisation X''' of X'' is given by six equations:

$$\begin{aligned}
\tilde{z}_1^2 + u\tilde{r}_1 &= 0\\
\tilde{z}_2^2 + u\tilde{r}_2 &= 0\\
t^2 - r_1 r_2 &= 0\\
ut - \tilde{z}_1\tilde{z}_2 &= 0\\
\tilde{z}_1 t + \tilde{z}_2\tilde{r}_1 &= 0\\
\tilde{z}_2 t + \tilde{z}_1\tilde{r}_2 &= 0
\end{aligned}$$

where we define a new variable $t = \frac{z_1 z_2}{y_1}$ and $\tilde{r}_1 = \tilde{a}_1 + \tilde{b}_1\tilde{z}_1 u^i + \tilde{c}_1\tilde{z}_2 u^j + \tilde{d}_1\tilde{z}_1\tilde{z}_2 u^{i+j}$, and $\tilde{r}_2 = \tilde{a}_2 + \tilde{b}_2\tilde{z}_1 u^i + \tilde{c}_2\tilde{z}_2 u^j + \tilde{d}_2\tilde{z}_1\tilde{z}_2 u^{i+j}$, thus we want to know in which condition it becomes smooth or at least Gorenstein. We look at the Jacobian of $\tilde{X}$

$$\Delta = \begin{pmatrix}
2\tilde{z}_1 + u\partial_{\tilde{z}_1}\tilde{r}_1 & u\partial_{\tilde{z}_2}\tilde{r}_1 & 0 & \tilde{r}_1 + u\partial_u\tilde{r}_1 & u\partial_{\overline{v}}\tilde{r}_1\\
u\partial_{\tilde{z}_1}\tilde{r}_2 & 2\tilde{z}_2 + u\partial_{\tilde{z}_2}\tilde{r}_2 & 0 & \tilde{r}_2 + u\partial_u\tilde{r}_2 & u\partial_{\overline{v}}\tilde{r}_2\\
-\partial_{\tilde{z}_1}(\tilde{r}_1\tilde{r}_2) & -\partial_{\tilde{z}_2}(\tilde{r}_1\tilde{r}_2) & 2t & -\partial_u(\tilde{r}_1\tilde{r}_2) & -\partial_{\overline{v}}(\tilde{r}_1\tilde{r}_2)\\
-\tilde{z}_2 & -\tilde{z}_1 & u & t & 0\\
t + \tilde{z}_2\partial_{\tilde{z}_1}\tilde{r}_1 & \tilde{r}_1 + \tilde{z}_1\partial_{\tilde{z}_2}\tilde{r}_1 & \tilde{z}_1 & \tilde{z}_2\partial_u\tilde{r}_1 & \tilde{z}_2\partial_{\overline{v}}\tilde{r}_1\\
\tilde{r}_2 + \tilde{z}_1\partial_{\tilde{z}_1}\tilde{r}_2 & t + \tilde{z}_1\partial_{\tilde{z}_2}\tilde{r}_2 & \tilde{z}_2 & \tilde{z}_1\partial_u\tilde{r}_2 & z_1\partial_{\overline{v}}\tilde{r}_2
\end{pmatrix}$$

Over $u = \overline{v} = 0$ (implies $\tilde{z}_1 = \tilde{z}_2 = 0$) Δ becomes (after rearrange the rows)

$$\Delta = \begin{pmatrix}
0 & 0 & 0 & a_1 & 0\\
0 & 0 & 0 & a_2 & 0\\
0 & 0 & 0 & t & 0\\
t & a_1 & 0 & 0 & 0\\
a_2 & t & 0 & 0 & 0\\
-\partial_{\tilde{z}_1}(r_1 r_2) & -\partial_{\tilde{z}_2}(r_1 r_2) & 2t & -\partial_u(a_1 a_2) & -\partial_{\overline{v}}(a_1 a_2)
\end{pmatrix}$$

. By the Jacobian criterion, the singular locus is given by the vanishing of the non zero 3×3 minors:

$$\begin{aligned}
t \cdot (t^2 - a_1 a_2) &= 0\\
\partial_u(a_1 a_2) \cdot (t^2 - a_1 a_2) &= 0\\
\partial_{\overline{v}}(a_1 a_2) \cdot (t^2 - a_1 a_2) &= 0
\end{aligned}$$

$X'''_{Sing} = Z(\tilde{z}_1^2, \tilde{z}_2^2, t^2 - a_1 a_2, \Delta) = Z(\tilde{z}_1, \tilde{z}_2, t^2 - a_1 a_2)$. For the Gorenstein condition we see that over $(u, \overline{v}) = (0, 0)$ the central fibre is $Z(\tilde{z}_1^2, \tilde{z}_2^2, t^2 - a_1 a_2, \tilde{z}_1\tilde{z}_2, \tilde{z}_1 t + \tilde{z}_2 a_1, \tilde{z}_2^2 t + \tilde{z}_1 a_2)$. Therefore X''' is Gorenstein if and only if one of the $\tilde{z}_j t + \tilde{z}_i a_j$ is eliminated from a_i and that X''' is not Gorenstein along $u = 0$ if and only if $a_1(0,0) = a_2(0,0) = 0$. We have proved

Proposition 2.11 — *With these notation we have*

1. *Singular locus of X''' over $(0,0)$ is $Z(z_1, z_2, t^2 - a_1a_2)$,*
2. *X''' is Gorenstein if and only if $u \neq 0$ or $(r_1(0,\overline{v}), r_2(0,\overline{v})) = 1$.*

We refrain from taking the analysis of this case any further.

3. Strata of normal surfaces

We attack the strata parametrising normal surface by direct methods. The starting points are the numerical restrictions following from [FPR15b].

Lemma 3.1 — *[Ant18] Let X be a normal Gorenstein stable surface with $K^2 = 1$ and $\chi(\mathcal{O}_X) = 2$. Let $\varepsilon\colon \tilde{X} \to X$ be a minimal resolution and r be number of elliptic singularities of X. Then $\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_X) - r$.*

Theorem 3.2 — *Let X be a normal Gorenstein stable surface with $K^2 = 1$ and $\chi(X) = 2$. Let $\varepsilon\colon \tilde{X} \to X$ and $\eta\colon \tilde{X} \to \tilde{X}_{min}$ be a morphism to a minimal model.*

Then only the following cases can occur; we list some invariants in Table 5

$\kappa(\tilde{X}) = 2$ *In this case X has canonical singularities and $\tilde{X} = \tilde{X}_{min}$ is the corresponding minimal surface of general type.*

$\kappa(\tilde{X}) = 1$ *$\tilde{X} = \tilde{X}_{min}$ is a minimal properly elliptic surface and X has precisely one elliptic singularity of degree 1.*

$\kappa(\tilde{X}) = 0$ *There exists an effective nef divisor D_{min} on $\tilde{X}_{min}$ and a point P such that:*

1. *$D_{min}^2 = 2$ and $p \in D_{min}$ has multiplicity 2.*
2. *$\eta\colon \tilde{X} \to \tilde{X}_{min}$ is the blow up at P.*
3. *X is obtained from $\tilde{X}$ by blowing down the strict transform of D_{min} and it has either one elliptic singularity of degree 2 or two elliptic singularties of degree 1.*

$\kappa(\tilde{X}) = -\infty$ *There are two possibilities:*

1. *$\chi(\tilde{X}) = 1$ and $\tilde{X}$ has one elliptic singularity;*
2. *$\chi(\tilde{X}) = 0$, $\tilde{X}$ has two elliptic singularities; in this case, the exceptional divisors arising from the elliptic singularities are smooth elliptic curves.*

In both cases, the degree of the elliptic sigularities is bounded by 4.

Proof. This follows quite directly from [FPR15b, Theorem 4.1], combined with the Enriques classification of surfaces. Assume that X has r elliptic singularities. Then $\chi(X) - r = \chi(\tilde{X}) = \chi(\tilde{X}_{min})$, so we can identify the number of elliptic singularities required in each of the numerical case.

To bound the degree of the elliptic singularities, note that by the algebraic description (Section 1) the surface X is embedded as a local complete intersection of codimension two in smooth variety. By the classification of elliptic singularities [Rei97], or lci slc singularities (see [Tzi09, Lemma 2.6]) this excludes elliptic points of degree higher than 4. □

We will now consider some of these numerical cases in more detail.

Table 5: Strata of normal surfaces

$\kappa(\tilde{X})$	**Elliptic sing.** $(d_1, .., d_r)$	$\chi(\tilde{X})$	**type**	Reference
2		2	gen. type	
1	(1)	1	min. prop. ell	Section 3.1
0	(2)	1	Enriques	Section 3.2
	(1,1)	0	Torus case	Section 3.4
	(1,1)	0	bielliptic case	Section 3.3
$-\infty$	(d)	1	rational	Section 3.5
	(d_1, d_2)	0	ruled over ell. curve	Section 3.5

3.1. Surfaces with properly elliptic minimal resolution

We now consider the following situation: Let X be a normal Gorenstein stable surface with $K_X^2 = 1$ and $\chi(X) = 2$ such that its minimal resolution $\tilde{X}$ has Kodaira dimension 1. Then by 3.2 the surface $\tilde{X}$ is minimal properly elliptic and $\varepsilon\colon \tilde{X} \to X$ contracts a unique curve E, smooth elliptic or a cycle of rational curves, with $E^2 = -1$ and possibly some ADE configurations.

Lemma 3.3 — *Let $\pi\colon \tilde{X} \to B$ be the elliptic fibration on $\tilde{X}$. Consider the sheaf $R^1\pi_*\mathcal{O}_{\tilde{X}}$ and denote its dual by $L = R^1\pi_*\mathcal{O}_{\tilde{X}}{}^\vee = \pi_*\omega_{\tilde{X}/B}$. Then we have*

$$\varepsilon^* K_X = K_{\tilde{X}} + E, K_{\tilde{X}}^2 = 0, E^2 = -1, K_{\tilde{X}}E = 1, \chi(\tilde{X}) = 1, p_g(\tilde{X}) = q(\tilde{X}),$$

L is a line bundle on B of degree 1 and one of the following cases occurs

Type A $p_g(\tilde{X}) = q(\tilde{X}) = 0 = g(B)$

Type B $p_g(\tilde{X}) = q(\tilde{X}) = 1 = g(B)$

Moreover, in both cases we have $h^0(2K_{\tilde{X}}) = 2$.

Proof. First of all, the map $\varepsilon\colon \tilde{X} \to X$ contracts a unique curve E, we have $\varepsilon^*K_X = K_{\tilde{X}} + E$, $\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_X) - 1 = 1$ and $K_{\tilde{X}}^2 = K_X^2 - 1 = 0$. The rest follows from the intersection numbers. To estimate $p_g(\tilde{X})$ we use that we have an injection $H^0(\tilde{X}, K_{\tilde{X}}) \to H^0(\tilde{X}, \varepsilon^*K_X)$, thus $p_g(\tilde{X}) \le p_g(X) = 1$.

For the last statement note that

$$H^0(2K_{\tilde{X}}) = H^0(2\varepsilon^*K_X - 2E) \subset H^0(2\varepsilon^*K_X) = \varepsilon^*H^0(2K_X).$$

By Corollary 1.7, $|2K_X|$ defines a quadruple cover of $\mathbb{P}^2$. Let x_0 be the image of the elliptic singularity, that is, the image of E under the composition $\tilde{X} \to X \to \mathbb{P}^2$. Then the general line in $\mathbb{P}^2$ does not contain x_0, so $h^0(2K_{\tilde{X}}) \le 2$. On the other hand, if l is a line through x_0 then the pullback of l contains at least twice the exceptional divisor E, since the elliptic singularity is a double point. Hence $h^0(2K_{\tilde{X}}) = 2$ as claimed. □

Let us fix some notation: denote by F_i the reduced multiple fibres of π with multiplicities m_i. By the canonical bundle formula [Fri12, Thm. 15, Sect. 7] we have

$$K_{\tilde{X}} = \pi^*(K_B + L) + \sum_{i=1}^{r}(m_i - 1)F_i \tag{3.4}$$

Let p_i be the image of F_i in B. By [Fri12, Section 7, Exercise 2] we have

$$H^0(mK_{\tilde{X}}) = \pi^* H^0(B, m(K_B + L) + \sum_{i=1}^{r} \lfloor \frac{m(m_i - 1)}{m_i} \rfloor p_i). \tag{3.5}$$

3.1.1. Type A

In this case $g(B) = 0$ so $B \cong \mathbb{P}^1$. Plugging $\deg L = 1$ into (3.5) and using Lemma 3.3 we compute

$$\begin{aligned} 2 = h^0(2K_{\tilde{X}}) &= h^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(-2 + \sum_{i=1}^{r}\left\lfloor \frac{2(m_i - 1)}{m_i} \right\rfloor\right)\right) \\ &= h^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2 + r)\right) \\ &= 1 + r - 2. \end{aligned}$$

Thus there are exactly three multiple fibres.

Note that since π is minimal, the curve E cannot be contained in a fibre, so it is a k-multisection with $k \geq 2$, because E has arithmetic genus 1 and B has genus 0. Then $E.F_i = k/m_i$ and by (3.4) and Lemma 3.3 we have

$$\begin{aligned} 1 = K_{\tilde{X}}E &= \pi^*(K_B + L).E + \sum_{i=1}^{3}(m_i - 1)F_i.E \\ &= k\left(-1 + \sum_{i=1}^{3}\frac{m_i - 1}{m_i}\right) \\ &\geq 2\left(-1 + \sum_{i=1}^{3}\frac{m_i - 1}{m_i}\right). \end{aligned}$$

Clearly $(m_i - 1)/m_i \geq 1/2$ and thus the only possiblility is $k = m_1 = m_2 = m_3 = 2$. We have proved

Lemma 3.6 — *If $\pi\colon \tilde{X} \to B$ is of Type A then E is a bisection and π has exactly three double fibres with reductions F_1, F_2, F_3. In particular $F_iE = 1$.*

Lemma 3.7 — *There exist points $q_i \in F_i$ such that $q_i \notin E$ and $(K_{\tilde{X}/B} + E)|_{F_i}$ is linearly equivalent to q_i.*

Proof. Note that $\mathcal{O}_{F_i}(F_i)$ is a non-trivial 2-torsion bundle on F_i by [Fri12, Thm. 15, Sect. 7]. We have

$$K_{\tilde{X}/B} = K_{\tilde{X}} - \pi^*K_B = \pi^*L + \sum_j F_j,$$

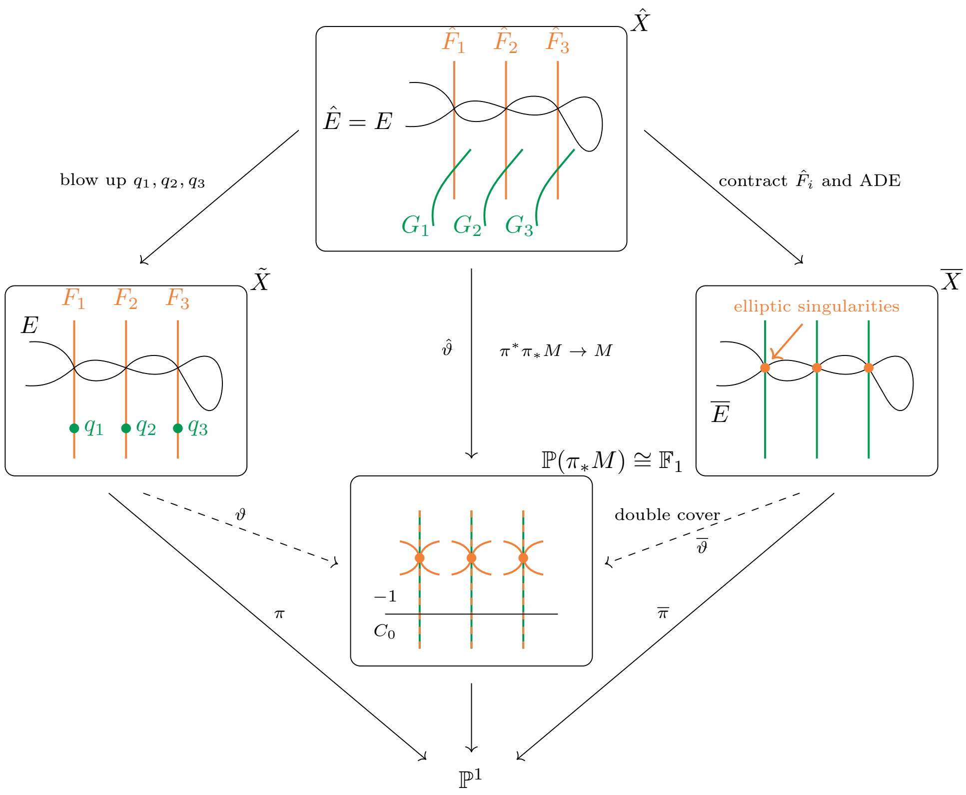


Figure 4: Properly elliptic case, type A

thus

$$(K_{\tilde{X}/B} + E)|_{F_i} = (\pi^* L + \sum_j F_j + E)|_{F_i} = (E + F_i)|_{F_i}$$

which has degree 1 and thus is linearly equivalent to a unique effective divisor q_i. We have $q_i \notin E$ because $F_i|_{F_i}$ is non-trivial. $\square$

Now we consider $\sigma\colon \hat{X} = \mathrm{Bl}_{q_1,q_2,q_3}(\tilde{X}) \to \tilde{X}$ and denote the exceptional curve over q_i with G_i. Let $\hat{\pi} = \pi \circ \sigma\colon \hat{X} \to B$ be the induced fibration.

Let $\hat{E}$ respectively $\hat{F}_i$ be the strict transforms of E and the F_i in $\hat{X}$. Let $\overline{\sigma}\colon \hat{X} \to \overline{X}$ be the contraction of the curves $\hat{F}_i$ and possibly of ADE-configurations in the singular fibres of $\hat{\pi}$, which do not intersect the bi-section $\hat{E}$.

Lemma 3.8 — *Consider on $\hat{X}$ the line bundle*

$$M = K_{\hat{X}/B} + \hat{E} - 2\sum_i G_i = \hat{\pi}^* L + \hat{E} + \sum_i \hat{F}_i. \tag{3.9}$$

Then the following properties hold:

1. $M|_{\hat{F}_i} \cong \mathcal{O}_{\hat{F}_i}$ *and* $\mathcal{O}_{\hat{F}_i}(\hat{F}_i) \cong \mathcal{O}_{\hat{F}_i}(-\hat{E})$*;*

2. *$M|_{n\hat{F}_i} \cong \mathcal{O}_{n\hat{F}_i}$ for every $n \in \mathbb{N}$ (and every $i = 1, 2, 3$);* [1]

3. *The sheaf $\overline{M} \cong \overline{\sigma}_* M$ is a line bundle and $M \cong \overline{\sigma}^* \overline{M}$;*

4. *For every (scheme-theoretic) fibre $\overline{X}_b$ of $\overline{\pi}\colon \overline{X} \to \mathbb{P}^1$ the line bundle $\overline{M}|_{\overline{X}_b}$ has two sections which define a base-point free pencil;*

5. *Then the natural map $\hat{\pi}^*\hat{\pi}_* M \to M$ is surjective and induces a morphism $\hat{\vartheta}\colon \hat{X} \to \mathbb{P}(\hat{\pi}_* M)$, which factors over $\overline{X}$ such that $\overline{\vartheta}\colon \overline{X} \to \mathbb{P}(\hat{\pi}_* M)$ is a double cover.*

In total the following diagram arises, compare also Figure 4:

$$\begin{tikzcd}
 & \hat{X} \arrow[dl, "\text{Blow up } q_1, q_2, q_3"'] \arrow[dr, "\text{contract } \hat{F}_i \text{ and ADE}"] \arrow[dd, "\hat{\vartheta}"] & \\
\tilde{X} \arrow[dr, dashed, "\vartheta"] \arrow[ddr, "\pi"'] & & \overline{X} \arrow[dl, "\overline{\vartheta}"', "\text{double cover}"] \arrow[ddl, "\overline{\pi}"] \\
 & \mathbb{P}(\pi_* M) \arrow[d] & \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

Proof. Denote by $\overline{G}_i = \overline{\sigma}(G_i)$ the image of G_i in $\overline{X}$ and also $\overline{E} = \overline{\sigma}(\hat{E})$. Note that $\hat{\pi}\colon \hat{X} \to \mathbb{P}^1$ factors over a map $\overline{\pi}\colon \overline{X} \to \mathbb{P}^1$.

1. For this item we use the first description of M, $M = K_{\hat{X}|B} + \hat{E} - \sum 2G_i$. By the blow up properties we have
$$K_{\hat{X}} = \sigma^* K_{\tilde{X}} + \sum G_i.$$
Thus
$$(K_{\hat{X}|B} + \hat{E} - \sum 2G_i) = \sigma^* K_{\tilde{X}|B} + E - \sum G_i.$$
The strict transform of F_i is $\hat{F}_i \cong F_i$. Then $(K_{X|B} + E)|_{F_i} = q_i$,
$$\begin{aligned} M|_{\hat{F}_i} =& (\sigma^*(K_{\tilde{X}|B} + E)|_{F_i} - \sum G_i)|_{\hat{F}_i} \\ =& q_i - q_i = 0. \end{aligned}$$
Thus $M|_{\hat{F}_i} = \mathcal{O}_{\hat{F}_i}$ and $\mathcal{O}_{\hat{F}_i}(\hat{F}_i) \cong \mathcal{O}_{\hat{F}_i}(-\hat{E})$.

2. We prove the assertion by induction on n. The $n = 1$ case is already done. We set $F := \hat{F}_i$ and consider the short exact sequence
$$0 \to \mathcal{O}_F\big(-(n-1)F\big) \to \mathcal{O}_{nF} \to \mathcal{O}_{(n-1)F} \to 0\,. \qquad (3.10)$$
Tensoring with M gives
$$0 \to \mathcal{O}_F\big(-(n-1)F\big) \to M_{|nF} \to \mathcal{O}_{(n-1)F} \to 0\,. \qquad (3.11)$$

[1] We are grateful to Andreas Krug for help with this item.

where the triviality of the outer terms is due to the induction hypothesis. In the following, we will show that $\mathsf{Ext}^1_{\mathcal{O}_{nF}}\Big(\mathcal{O}_{(n-1)F}, \mathcal{O}_F\big(-(n-1)F\big)\Big) = \mathbb{C}$, i.e., there are two non-trivial extensions. The assertion then follows by comparing the two exact sequences (3.10) and (3.11) (note that (3.11) cannot split since $M_{|nF}$ is a line bundle).

Let $\alpha\colon F \hookrightarrow (n-1)F$ and $\iota\colon (n-1)F \hookrightarrow nF$ be the closed embeddings. We are going to apply the language of derival bundle to compute $\mathsf{Ext}^1_{\mathcal{O}_{nF}}\Big(\mathcal{O}_{(n-1)F}, \mathcal{O}_F\big(-(n-1)F\big)\Big)$. We have $(L\iota^*)\iota_*\mathcal{O}_{(n-1)F} \cong \mathcal{O}_{(n-1)F}[0] \oplus \mathcal{O}_{(n-1)F}(-(n-1)F)[1]$; see [AC12, Thm. 0.7] and note that $\mathcal{O}_{(n-1)F}(-(n-1)F)$ is the conormal bundle of the embedding ι. This gives

$$
\begin{aligned}
&\mathsf{Ext}^1_{\mathcal{O}_{nF}}\Big(\iota_*\mathcal{O}_{(n-1)F}, \iota_*\alpha_*\mathcal{O}_F\big(-(n-1)F\big)\Big) \\
&\cong \mathsf{Ext}^1_{\mathcal{O}_{(n-1)F}}\Big((L\iota^*)\iota_*\mathcal{O}_{(n-1)F}, \alpha_*\mathcal{O}_F\big(-(n-1)F\big)\Big) \\
&\cong \mathsf{Ext}^1_{\mathcal{O}_{(n-1)F}}\Big(\mathcal{O}_{(n-1)F}, \alpha_*\mathcal{O}_F\big(-(n-1)F\big)\Big) \\
&\qquad \oplus \mathsf{Ext}^0_{\mathcal{O}_{(n-1)F}}\Big(\mathcal{O}_{(n-1)F}(-(n-1)F), \alpha_*\mathcal{O}_F\big(-(n-1)F\big)\Big) \\
&\cong H^1\big(F, \mathcal{O}_F(-(n-1)F)\big) \oplus H^0\big(F, \mathcal{O}_F\big) \\
&\cong \mathbb{C}\,.
\end{aligned}
$$

3. Let $p_i \in \overline{X}$ be the image of $\hat{F}_i$ in $\overline{X}$. By [Eis13, Exercise 7.5] it is enough to prove that the completion of $\overline{\sigma}^*M$ at p_i is free.

   By the theorem of formal functions [Har77, Thm. III.11.1] combined with the fact that the chosen subscheme structure on the fibre does not affect the limit [Har77, Rem. II.9.3.1] we have

$$
\widehat{\overline{\sigma}_*M}^{p_i} \cong \varprojlim_n H^0(nF_i, M|_{nF_i}) \cong \varprojlim_n H^0(nF_i, \mathcal{O}_{nF_i}) \cong \widehat{\overline{\sigma}_*\mathcal{O}_{\hat{X}}}^{p_i}, \tag{3.12}
$$

   where the middle isomorphism comes from the previous item.

   Since the contraction of an elliptic curve with self-intersection -1 leads to a hypersurface singularity [Rei97, Ch. 4] we have $\overline{\sigma}_*\mathcal{O}_{\hat{X}} \cong \mathcal{O}_{\overline{X}}$. So indeed the right hand side of (3.12) is free.

4. This is easily computed on the fibres of the form $2\overline{G}_i$ and clear on the general fibre.

5. By base change and the previous step $\hat{\pi}_*M = \overline{\pi}_*\hat{M}$ is a vector bundle of rank 2 and $\overline{\pi}^*\overline{\pi}_*\overline{M} \to \overline{M}$ is surjective because it is fibrewise base-point free. Fibrewise $\overline{X} \to \mathbb{P}(\hat{\pi}_*M)$ is a double cover, so it is a double cover. $\square$

Lemma 3.13 — *We have $\hat{\pi}_*M \cong \mathcal{O}_B(1) \oplus \mathcal{O}_B(2)$ and thus $P = \mathbb{P}(\hat{\pi}_*M) \cong \mathbb{F}_1$. Let C_0 be the unique (-1)-curve in P and F a general fibre of the projection to $B = \mathbb{P}^1$. Then the tautological bundle of P is $\mathcal{O}_P(1) = \mathcal{O}_P(C_0 + 2F)$ and $\vartheta_*\hat{E}$ is an irreducible curve in $|C_0 + F|$.*

Proof. First note that for each i, $\sigma^* F_i = \hat{F}_i + G_i$, thus

$$\sigma_* \mathcal{O}_{\hat{X}}(\sum_i \hat{F}_i) = \sigma_* \mathcal{O}_{\hat{X}}(\sum_i \sigma^* F_i - G_i) = \mathcal{I}_{\{q_1,q_2,q_3\}}(\sum_i F_i) \subset \mathcal{O}_{\tilde{X}}(\sum_i F_i),$$

so that by [Fri12, Ch.7, Exercise 2] we have

$$\hat{\pi}_* \mathcal{O}_{\hat{X}}(\sum_i \hat{F}_i) \subset \pi_* \mathcal{O}_{\tilde{X}}(\sum_i F_i) = \mathcal{O}_B.$$

Since the left hand side has a global section, this inclusion is an equality. This implies that the pushforward $\hat{\pi}_*(\hat{\pi}^* L + \sum_i \hat{F}_i) = L$. Now consider the exact sequence

$$0 \to \hat{\pi}^* L + \sum_i \hat{F}_i \to M \to M|_{\hat{E}} = K_{\hat{E}/B} \to 0.$$

Applying $\hat{\pi}_*$ we get by the projection formula, the above computation and using both descriptions of M from (3.9),

$$\begin{aligned} 0 \to L = \hat{\pi}_*(\hat{\pi}^* L + \sum_i \hat{F}_i) \to \hat{\pi}_* M \to \hat{\pi}_* K_{\hat{E}/B} = (\hat{\pi}_* \mathcal{O}_{\hat{E}})^\vee \to \\ R^1 \hat{\pi}_*(K_{\hat{X}/B} - \sum_i 2G_i) \to R^1 \hat{\pi}_* M \to ... \end{aligned} \tag{3.14}$$

By relative duality we have

$$R^1 \hat{\pi}_*(K_{\hat{X}/B} - \sum_i 2G_i) = R^1 \hat{\pi}_* \mathscr{H}\mathrm{om}(\sum_i 2G_i, K_{\hat{X}/B}) = \mathscr{H}\mathrm{om}(\hat{\pi}_*(\sum_i 2G_i), \mathcal{O}_B) \cong \mathcal{O}_B,$$

where the last identification is proved by pushing forward the exact sequence

$$0 \to \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\hat{X}}(\sum_i 2G_i) \to \mathcal{O}_{\sum_i 2G_i}(\sum_i 2G_i) \to 0.$$

Indeed, $\hat{\pi}_* \mathcal{O}_{\sum_i 2G_i}(\sum_i 2G_i)$ is a sum of skyscraper sheaves supported at the images of the G_i with stalks $H^0(2G_i, \mathcal{O}_{2G_i}(2G_i))$. These are zero, because $G_i^2 = -1$. The assertion is deduced from

$$0 \to \hat{\pi}_* \mathcal{O}_{\hat{X}} = \mathcal{O}_B \to \hat{\pi}_* \mathcal{O}_{\hat{X}}(\sum_i 2G_i) \to 0.$$

Repeating this for $M = K_{\hat{X}/B} - \sum_i 2G_i + \hat{E}$ we get

$$R^1 \hat{\pi}_* M = \left(\hat{\pi}_* \mathcal{O}_{\hat{X}}(\sum_i 2G_i - \hat{E}) \right)^\vee = 0$$

because $\mathcal{O}_{\hat{X}}(\sum_i 2G_i - \hat{E})$ restricted to the general fibre has negative degree and thus no sections, so $\hat{\pi}_* \mathcal{O}_{\hat{X}}(\sum_i 2G_i - \hat{E})$ is a torsion sheaf and its dual is trivial.

Therefore the sequence (3.14) is isomorphic to

$$0 \to \mathcal{O}_B(1) \to \hat{\pi}_* M \to \mathcal{O}_B \oplus \mathcal{O}_B(2) \to \mathcal{O}_B \to 0,$$

and since $\mathsf{Hom}(\mathcal{O}_B(2), \mathcal{O}_B) = 0$ and $\mathsf{Ext}^1(\mathcal{O}_B(2), \mathcal{O}_B(1)) = 0$ we have $\hat{\pi}_* M \cong \mathcal{O}(1) \oplus \mathcal{O}_B(2)$.

We now use the theory of ruled surfaces, compare [Har77, Section V]. Let $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_B(-1) \oplus \mathcal{O}_B)$ so that the relative tautological bundle is $\mathcal{O}_{\mathbb{F}_1}(1) = \mathcal{O}_{\mathbb{F}_1}(C_0)$. Since $\hat{\pi}_* M = (\mathcal{O}_B(-1) \oplus \mathcal{O}_B) \otimes \mathcal{O}_B(2)$ we have $\mathcal{O}_P(1) = C_0 + 2F$.

Since $\overline{X} \to P$ is a double cover $\hat{\vartheta}(\hat{E})$ is a section of the projective bundle, that is, is an irreducible curve in a linear system $|C_0 + kF|$ for some k. To determine k we compute by the projection formula

$$\begin{aligned} k - 1 = (kF + C_0)C_0 = \hat{\vartheta}_* \hat{E} C_0 = 2\hat{E}.\hat{\vartheta}^* C_0 \\ = 2\hat{E}.\hat{\vartheta}^*(\mathcal{O}_P(1) - 2F) = 2\hat{E}.(M - \pi^* \mathcal{O}_B(2)), \end{aligned}$$

By the defintion of M and the projection formula we get $\hat{E}.(M - \pi^* \mathcal{O}_B(2)) = 0$, thus $k = 1$ as claimed. □

We now determine the geometry and class of the ramification divisor $R \subset P$ of the double cover $\overline{\vartheta} \colon \overline{X} \to P$. Since the general fibre of $\hat{\pi}$ is an elliptic curve, the ramification divisor intersects the general fibre P in four points and we have $R \sim 4C_0 + kF$ for some k.

Now to determine k we write $\overline{\vartheta}_* \mathcal{O}_{\overline{X}} \cong \mathcal{O}_P \oplus \mathscr{L}^{-1}$ so that $R \in |2\mathscr{L}| = |2(2C_0 + k/2F)|$ and compute

$$1 = \chi(\tilde{X}) = \chi(\hat{X}) = \chi(\overline{X}) - 3 = \chi(\overline{\vartheta}_* \mathcal{O}_{\overline{X}}) - 3.$$

This implies $\chi(\mathscr{L}^{-1}) = 3$ and by Riemann Roch since $K_P = -2C_0 - 3F$ we have

$$3 = \chi(\mathscr{L}^{-1}) = 1 + \frac{1}{2}\left(-2C_0 - \frac{k}{2}F\right)\left(-2C_0 - \frac{k}{2}F + 2C_0 + 3F\right) \Rightarrow k = 10.$$

In other words $R \in |4C_0 + 10F|$. We can say more about the ramification R of the double cover $\overline{\vartheta}$: First of all $\overline{\pi} \colon \overline{X} \to B$ has exactly three double fibres while P has none, so we can write $R = \tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{C}$ for a curve $C \in |7F + 4C_0|$ and the three fibres of $P \to B$ sitting over p_1, p_2, p_3.

Note that $\overline{X}$ has three elliptic singularities of degree 1 and that image of $\hat{E}$ passes through all three of them. In addition, these elliptic singularities are contained in double fibres of $\overline{\pi} \colon \overline{X} \to B$. By the classification of singularities of double covers [FPR17, Ant18] this means that R has possibly degenerate $[3, 3]$ points at the intersection points $\tilde{L}_i \cap \hat{\vartheta}(\hat{E})$ and ADE singularities elsewhere, because $\overline{X}$ has no further non-canonical singularities.

Note that C_0 and E are disjoint sections, because all irreducible curves in $|C_0 + F|$ do not meet C_0.

Now let $\alpha \colon P \to \mathbb{P}^2$ be the blow down of C_0. Then $\alpha(\hat{\vartheta}(\hat{E}))$ is a line in $\mathbb{P}^2$, disjoint from the point we blow up. We may choose coordinates such that

- C_0 maps to $p = (0 : 0 : 1)$,
- $\alpha(\hat{\vartheta}(\hat{E}))$ is the line $\{z = 0\}$,
- $\alpha(\tilde{L}_1) = L_1 = \{x = 0\}$,

- $\alpha(\tilde{L}_2) = L_2 = \{y = 0\}$,
- $\alpha(\tilde{L}_3) = L_3 = \{x - y = 0\}$,

Write $4C_0 + 7F = 7(C_0 + F) - 3C_0 = 7\hat{\vartheta}_*\hat{E} - 3C_0 = 7\alpha^*\mathcal{O}_{\mathbb{P}^2}(1) - 3C_0$ we have that

$$\begin{aligned} H^0(P, 4C_0 + 7F) &\cong H^0(P, 7\alpha^*\mathcal{O}_{\mathbb{P}^2}(1) - 3C_0) \\ &\cong H^0(\mathbb{P}^2, \alpha_*(\alpha^*\mathcal{O}_{\mathbb{P}^2}(7) - 3C_0)) \\ &\cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(7) \otimes \alpha_*(\mathcal{O}_P(-3C_0))) \\ &\cong H^0(\mathbb{P}^2, \mathcal{I}_p^3(7)), \end{aligned}$$

and collecting the information from above we see that $\alpha(\tilde{C}) = C$ is a plane septic with the following properties

1. C has at least a triple point at p but $\alpha^*C - 3C_0$ has ADE singularities near C_0,
2. C has (possibly degenerate) $[2,2]$ points at $(1:0:0)$, $(0:1:0)$, and $(1:1:0)$ which are tangent to the L_i, that is, $C + \sum_i L_I$ has three $[3,3]$ points at these points,
3. elsewhere C has at most ADE singularities.

Going backwards we have proved.

Proposition 3.15 — *Every surface of type A arise from a plane septic via the above description.*

Example 3.16 — Let D_1 be a union of three general lines through D_2. The result by Theorem 3.2 is a surface with a minimal properly elliptic surface, see Figure 5. In the Figure 5, the three fibres F_i are the three lines D_1, and the three points q_i are intersection points of D_0 and D_1. Indeed, let P be the intersection point of the three general lines on D_1. The bi-canonical section of $\tilde{X}$ after blowing up at P is $2K_{\tilde{X}} = 2\pi^*K_X - 2E$ which vanishes twice along E. It must be the pencil through the point P and thus the q_i must be the intersection points of the D_0 and D_1.

3.1.2. Type B

Proposition 3.17 — *If $\pi\colon \tilde{X} \to B$ is as in case B of Lemma 3.3 then B is an elliptic curve, E is a section, hence smooth elliptic, $K_{\tilde{X}} = \pi^*L$.*

Proof. Again E cannot be contained in a fibre since the π is a minimal elliptic fibration, so E is a k-multisection. If we apply the canonical bundle formula (3.4) with trivial K_B then we get

$$1 = E.K_{\tilde{X}} = E.\pi^*L + \sum_i (m_i - 1)F_i.E = k + \sum_i (m_i - 1)$$

and hence $k = 1$ and no multiple fibres. □

Remark 3.18 — Surfaces of type B have Weierstrass models as in [Fri12, Sect. 7, Thm. 20] and are thus parametrised by an irreducible family.

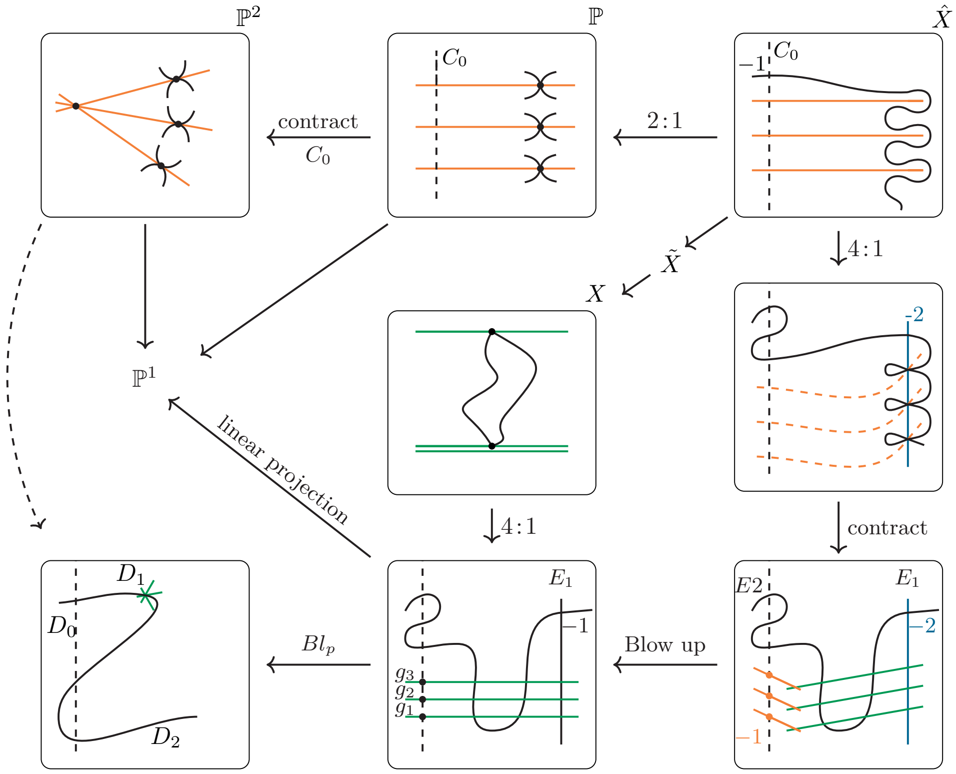


Figure 5: Surface with properly elliptic minimal resolution, Type A

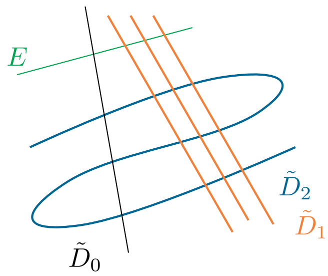


Figure 6: Surface with properly elliptic minimal resolution, Type B

Example 3.19 — Let D_1 be a union of three general lines through $P \in D_0$ and D_2 a general cubic. Then X has a unique elliptic singularity of degree 1. Blowing up at P and then changing the branch divisor to get a normal bi-double cover $\tilde{X} \to \mathbb{P}^2$. One compute that $|2K_{\tilde{X}}|$ is an elliptic pencil, induced by the pencil of lines passing through P. Thus $\tilde{X}$ is a minimal properly elliptic surface, see Figure 6.

3.2. Enriques case

In this case, X has a unique elliptic singularity of degree 2, the minimal resolution $\tilde{X}$ is an Enriques surface blown up in one point, which is a node on a nodal curve of genus 2. Thus, at least as a set, an open subset of this stratum is in bijection to the isomorphism classes of pairs

$$\mathcal{E} = \{(Y, C) \mid Y \text{ Enriques surface}, C \text{ nodal, ample curve}, p_a(C) = 2\}.$$

This suggests to study this stratum via the presumably finite and dominant map to the moduli space of Enriques surfaces. While quite some information is available on the latter (see e.g. [CD89, GH16]), the construction an exploration of $\mathcal{E}$ goes beyond the scope of this thesis.

Note however, that $\mathcal{E}$ is non-empty: either one can argue that in the linear system associated to a degree 2 polarisation on an Enriques surface not every member can be smooth for topological reasons, so in general there is a curve with just one node and arithmetic genus 2. Alternatively, an explicit example of a stable surface in this stratum was constructed in [FPR17, Example Z_2^E].

3.3. Bielliptic surface case

In this case, $\tilde{X}$ has two elliptic singularities of degree 1 and its minimal resolution $\tilde{X}$ has Kodaira dimension 0 and as minimal model a bi-elliptic surface.

We quickly recall the classification of bi-elliptic surfaces, which are the surfaces of Kodaira dimension 0 with $\chi(\tilde{X}) = 0 = p_g(\tilde{X})$.

Let A, B be elliptic curve and G a finite group acting on A by translations and on B such that $B/G \cong \mathbb{P}^1$. Then $\tilde{X}$ is of the form $\tilde{X} = A \times B/G$ and the possible

Table 6: Classification of bielliptic surfaces

Type of a bielliptic surface	G	$m_1, ..., m_s$	Basis of Num(S)
1	$\mathbb{Z}_2$	2, 2, 2, 2	$A/2, B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2, 2, 2, 2	$A/2, B/2$
3	$\mathbb{Z}_4$	2, 4, 4	$A/4, B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	2, 4, 4	$A/4, B/2$
5	$\mathbb{Z}_3$	3, 3, 3	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	3, 3, 3	$A/3, B/3$
7	$\mathbb{Z}_6$	2, 3, 6	$A/6, B$

cases where classified by Bagnera and de Franchis [BDF07], see Table 6. With the notation in the Table we denote $\mu = \mathrm{lcm}\{m_1, ..., m_s\}$ and $\gamma =$ order of G, where m_i is the multiplicity of the multiple fibre of $\pi_2 \colon \tilde{X} \to B/G \cong \mathbb{P}^1$. Note that a basis of $\mathrm{Num}(S)$ consists of divisors A/μ and $(\mu/\gamma)B$ where $A^2 = 0, B^2 = 0, AB = \gamma$.

Lemma 3.20 — *An elliptic curve E is contained in $\tilde{X}$ if numerically E is a positive multiple of A/μ or $(\mu/\gamma)B$.*

Proof. By the adjunction Formula, we have $2g(E) - 2 = E(E + K_{\tilde{X}})$. The canonical divisor $K_{\tilde{X}}$ of each bielliptic surface is numerically trivial. It follows that $E^2 = 0$, which happens only when E is a multiple of A/μ or $(\mu/\gamma)B$; it has to be positive, since $A.E \geq 0$ and $B.E \geq 0$. □

We get two fibrations on $\tilde{X}$, namely,

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A/G & & B/G \end{tikzcd}$$

By Theorem 3.2 we are looking for two elliptic curves E_1, E_2, neccesarily smooth, because $\tilde{X}$ does not contain rational curves, such that $E_1.E_2 = 1$.

Lemma 3.21 — *Such a configuration exists if and only if $\tilde{X}$ is an odd bielliptic surface, that is, in cases 1,3,5,7 in Table 6.*

Moreover, in this case up to automorphism $E_2 = B = \pi_1^{-1}(0)$ and E_1 is the reduction of a multiple fibre of maximal multiplicity of π_2.

Proof. Assume there are two elliptic curves E_1, E_2 on $\tilde{X}$ such that $E_1E_2 = 1$. By Lemma 3.20 we can write $E_1 = aA/\mu$, $E_2 = b(\mu/\gamma)B$ for some positive integers a, b. Then

$$1 = E_1E_2 = \frac{ab}{\gamma}AB = ab,$$

so we have $a = b = 1$. By [Far16, Lemma 2.7] we get $\mu/\gamma = 1$, because numerically $\mu/\gamma B = E_2$ is effective. This happens exactly in the cases $1, 3, 5, 7$ in the Table 6 as in [BHPV04] and the the only curves in these numerical classes are are the given ones. □

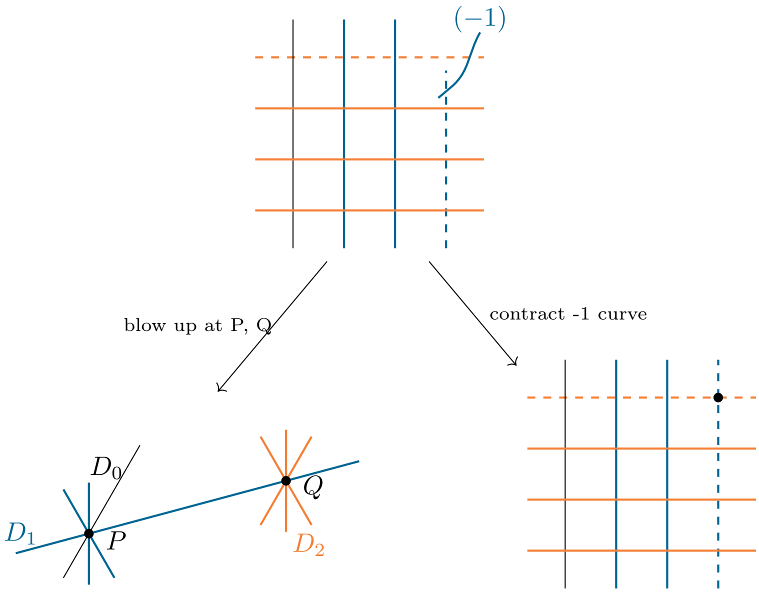


Figure 7: Torus case

Proposition 3.22 — *There are three* 1*-dimensional and one* 2*-dimensional strata of normal Gorenstein stable surfaces with $K_X^2 = 1$ and $\chi(X) = 2$ with minimal resolution birational to a bi-elliptic surface.*

Proof. Note that in all cases in the table the elliptic curve A is arbitrary. In case 1, the curve B can also be arbitrary, but in the other cases B admits a larger group of automorphisms, hence is isomorphic to $\mathbb{C}/\mathbb{Z}[i]$ or $\mathbb{C}/\mathbb{Z}[\exp(2\pi i/3)]$. So the number of parameters is two in the first case and 2 in the other cases. $\square$

Example 3.23 — This example comes from [FPR17]. Let D_1 be a union of three general lines through $P \in D_0$ and D_2 be a union of three lines passing through a general point $Q \in D_1$, see Figure 7. Then X has two elliptic singularities of degree 1. Blowing up at P and Q we get $\tilde{D_0}$ a (-1)-curve. Contracting this -1 curve and consider the bi double cover branched over we get a surface S over $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\varphi\colon S \to \mathbb{P}^1 \times \mathbb{P}^1$, we take into account that $\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \sum_{i=1}^3 L_i^{-1}$ where

$$L_1 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\frac{2}{2}, \frac{4}{2})$$
$$L_2 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2, 0)$$
$$L_3 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 2)$$

we conclude that $h^1(\mathcal{O}_S) = 1$ and thus S is a bielliptic surface.

This explicit example varies in a two-dimensional family, because the crossratio in the points where four lines meet can be arbitrary. Thus it gives the family of surfaces of type 1 in Table 6.

3.4. Torus case

According to Theorem 3.2, the remaining case in Kodaira dimension 0 is when the minimal model $\sigma\colon \tilde{X} \to \tilde{X}_{\min}$ of the minimal resolution is a torus.

Proposition 3.24 — *Let X be a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(X) = 2$ such that the minimal model of a resolution is a torus. Then there exist elliptic curves E_1, E_2 and a commutative diagram*

$$\begin{tikzcd}
 & \tilde{X} \arrow[dl, "\text{blow down } \hat{E}_1, \hat{E}_2"', "\varepsilon"] \arrow[dr, "\eta"] & \\
X \arrow[d, "\varphi"'] & (\mathbb{Z}/2)^2\text{-covers} & E_1 \times E_2 \arrow[d, "p_1 \times p_2"] \\
\mathbb{P}^2 \arrow[rr, dashed, "\text{projections}", "\text{from } P \text{ and } Q"'] & & \mathbb{P}^1 \times \mathbb{P}^1
\end{tikzcd}$$

such that

1. *the bicanonical map φ is a bi-double cover with building data a line D_0 containing two points P and Q, D_1 three lines through P and D_2 three lines through Q.*

2. *$p_i\colon E_i \to \mathbb{P}^1$ is the natural double cover.*

3. *η is the blow up of the intersection of point of $E_1 \cup E_2 \subset E_1 \times E_2$.*

4. *ε is the contraction of the strict transform of $E_1 \cup E_2$.*

Figure 8 depicts the change of building data under resolution of the birational transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Let $\varepsilon\colon \tilde{X} \to X$ be the minimal resolution and $\eta\colon \tilde{X} \to \tilde{X}_{\min}$ be a map to a minimal model. By Theorem 3.2 and our assumptions, $\tilde{X}_{\min}$ is a torus contining two elliptic curves E_1 and E_2 such that $E_1E_2 = 1$. Using the intersection point as origin for everything, the addition map $E_1 \times E_2 \to \tilde{X}_{\min}$ is an isomorphism.

We have proved the properties attributed to the upper part of the diagram.

Denoting the exceptional curve of η with F we have

$$\begin{aligned}
\varepsilon^* K_X &= K_{\tilde{X}} + \hat{E}_1 + \hat{E}_2 \\
&= \eta^* K_{E_1 \times E_2} + F + \hat{E}_1 + \hat{E}_2 \\
&= \eta^*(E_1 + E_2) - F
\end{aligned}$$

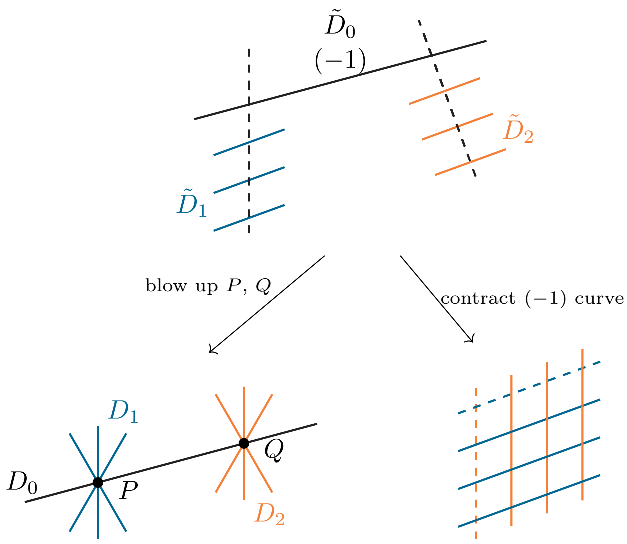


Figure 8: building data in the Torus case

and thus

$$
\begin{aligned}
H^0(X, 2K_X) &\cong H^0(\tilde{X}, \varepsilon^* 2K_X) \\
&= H^0(\tilde{X}, 2\eta^*(E_1 + E_2) - 2F) \\
&\cong H^0(E_1 \times E_2, \mathcal{I}^2_{E_1 \cap E_2}(2E_1 + 2E_2)) \\
&\subset H^0(E_1 \times E_2, 2E_1 + 2E_2) \\
&= H^0(E_1, p_1^* \mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(E_2, p_2^* \mathcal{O}_{\mathbb{P}^1}(1)) \\
&= p_1^* H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^* H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \\
&= \langle x_1, y_1 \rangle \otimes \langle x_2, y_2 \rangle.
\end{aligned}
$$

where we choose the sections such that x_i cuts out the divisor $2E_i$. With these coordinates and identifications, the bicanonical map of X is defined by the sections $\langle x_1 \otimes x_2, x_1 \otimes y_2, y_1 \otimes x_2 \rangle$. On $\mathbb{P}^1 \times \mathbb{P}^1$; these sections define exactly the inverse of the lower horizontal map in the diagram.

Thus the diagram commutes and all remaining claims follow easily. □

3.5. Minimal resolution of Kodaira dimension $-\infty$

In the case where the minimal resolution has Kodaira dimension $-\infty$ the situation is quite complicated, and we refrain from a detailed study.

From Theorem 3.2 we see that there is a finite number of possible cases and we have seen an example of surface with a single elliptic singularity of degree 4, constructed as a bi-double cover already in [FPR17].

Our hope to make progress on this case using the methods of Section 2 was squashed as well.

4. Strata of non-normal surfaces

To identify the non-normal Gorenstein stable surfaces with $K_X^2 = 1$ and $\chi(X) = 2$ we closely follow the strategy from [FPR18].

4.1. Normalisation and glueing: starting point of the classification

Let X be a non-normal stable surface and $\pi\colon \overline{X} \to X$ its normalisation. Recall that the non-normal locus $D \subset X$ and its preimage $\overline{D} \subset \overline{X}$ are pure of codimension 1, that is, curves. Since X has ordinary double points at the generic points of D the map on normalisations $\overline{D}^\nu \to D^\nu$ is the quotient by an involution τ. Kollár's glueing principle says that X can be uniquely reconstructed from $(\overline{X}, \overline{D}, \tau\colon \overline{D}^\nu \to \overline{D}^\nu)$ via the following two push-out squares:

$$\begin{tikzcd}
\overline{X} \arrow[d, "\pi"] & \overline{D} \arrow[l, hook', "\overline{\iota}"'] \arrow[d, "\pi"] & \overline{D}^\nu \arrow[l, "\overline{\nu}"'] \arrow[d, "/\tau"] \\
X & D \arrow[l, hook', "\iota"'] & D^\nu \arrow[l, "\nu"']
\end{tikzcd} \tag{4.1}$$

Applying this principle to non-normal Gorenstein stable surfaces, we deduce by [Kol13, Thm. 5.13] and [FPR15b, Addendum in Sect.3.1.2] that a triple $(\overline{X}, \overline{D}, \tau)$ corresponds to a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(X) = 2$ if and only if the following four conditions are satisfied:

lc pair condition $(\overline{X}, \overline{D})$ is an lc pair, such that $K_{\overline{X}} + \overline{D}$ is an ample Cartier divisor.

K_X^2**-condition** $(K_{\overline{X}} + \overline{D})^2 = 1$.

Gorenstein-glueing condition $\tau\colon \overline{D}^\nu \to \overline{D}^\nu$ is an involution that restricts to a fixed-point free involution on the preimages of the nodes of $\overline{D}$.

χ**-condition** The holomorphic Euler-characteristic of the non-normal locus D is $\chi(D) = 2 - \chi(\overline{X}) + \chi(\overline{D})$.

In [FPR15b] Gorenstein log canonical pairs $(\overline{X}, \overline{D})$ with $(K_{\overline{X}} + \overline{D})^2 = 1$ were classified:

(P) $\overline{X} = \mathbb{P}^2$ and $\overline{D}$ is a nodal quartic. Here $p_a(\overline{D}) = 3$ and $K_{\overline{X}} + \overline{D} = \mathcal{O}_{\mathbb{P}^2}(1)$.

(dP) $\overline{X}$ is a (possibly singular) Del Pezzo surface of degree 1, namely $\overline{X}$ has at most canonical singularities, $-K_{\overline{X}}$ is ample and $K_{\overline{X}}^2 = 1$. The curve $\overline{D}$ belong to the system $|-2K_{\overline{X}}|$, hence $K_{\overline{X}} + \overline{D} = -K_{\overline{X}}$ and $p_a(\overline{D}) = 2$.

(E_-) Let E be an elliptic curve and let $a\colon \tilde{X} \to E$ be a geometrically ruled surface that contains an irreducible section C_0 with $C_0^2 = -1$. Namely, $\tilde{X} = \mathbb{P}(\mathcal{O}_E + \mathcal{O}_E(-x))$, where $x \in E$ is a point and C_0 is the only one curve on the system $|\mathcal{O}_{\overline{X}}(1)|$. Set $F = a^{-1}(x)$: the normal surface $\overline{X}$ is obtained from $\tilde{X}$ by contracting C_0 to an elliptic Gorenstein singularity of degree 1 and $\overline{D}$ is the image of a curve $\overline{D}_0 \in |c(C_0 + F)|$ disjoint from C_0, so $p_a(\overline{D}) = 2$. The line bundle $K_{\overline{X}} + \overline{D}$ pulls back to $C_0 + F$ on $\tilde{X}$.

(E_+) $\overline{X} = S^2E$, where E is an elliptic curve. Let $a\colon \overline{X} \to E$ be the Albanese map, which is induced by the addition map $E \times E \to E$, denote by F the class of a fibre of a and by C_0 the image in $\overline{X}$ of the curve $\{0\} \times E + E \times \{0\}$, where $0 \in E$ is the origin, so that $C_0F = C_0^2 = 1$. Then $\overline{D}$ is a divisor numerically equivalent to $3C_0 - F, p_a(\overline{D}) = 2$ and $K_{\overline{X}} + \overline{D}$ is numerically equivalent to C_0.

4.2. Case (P)

It turns out that the classification of this case entails a detailed study of some plane quartics. For future reference we begin with an elementary lemma.

Lemma 4.2 — *Let $\overline{D}$ be a nodal plane quartic.*

1. *If $\overline{D}$ has at most two nodes, then it is irreducible.*

2. *If $\overline{D}$ has three nodes, then it is reducible if and only if the nodes are colinear if and only if it is the union of a smooth cubic and a general line.*

3. *If $\overline{D}$ has four nodes, then it is the union of two smooth conics or the union of a nodal cubic and a line.*

4. *If $\overline{D}$ has five nodes, then it is the union of a smooth conic and two general lines.*

5. *If $\overline{D}$ has at least six nodes, then it has exactly six nodes and is the union of four lines in general position.*

Proof. All statements are elementary by Bézout. Let us only point out that three colinear nodes force the line through the nodes to be contained in $\overline{D}$. If there are four nodes, then in the pencil of conics through the nodes there is at least one that intersects $\overline{D}$ with multiplicity larger than 8, and is thus contained in $\overline{D}$. □

Let us now set up the notation. For the whole section we consider a Gorenstein stable surface X with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$. With notation as in (4.1) the normalisation $\overline{X} = \mathbb{P}^2$ and $\overline{D}$ is a nodal quartic. Denote by

- μ_1, the number of degenerate cusps in X.
- ρ, the number of ramification points of the map $\overline{D}^\nu \to D^\nu$
- $\overline{\mu}$, the number of nodes of $\overline{D}$.

Remark 4.3 — For further use, note that the points of D^ν correspond to equivalence classes of points on $\overline{D}^\nu$ with respect to the relation generated by $x \sim y$ if $\overline{\nu}(x) = \overline{\nu}(y)$ or $\tau(x) = y$. By the classification of Gorenstein semi-log-canonical singularities (see the proof of Lemma 3.5 in [FPR15b]) nodes of $\overline{D}$ map to degenerate cusp singularities of X and preimages of degenerate cusp singularities are nodes of $\overline{D}$. Thus, the number μ_1 of degenerate cusps in X equals the number of equivalence classes of preimages of nodes in $\overline{D}^\nu$ under the above relation.

In our situation, by the χ-condition and [FPR15b, Lem. 3.5], we have the equality

$$-1 = \chi(D) = \frac{1}{2}\left(\chi(\overline{D}) - \overline{\mu}\right) + \frac{\rho}{4} + \mu_1,$$

which gives

$$\overline{\mu} = \mu_1 = \rho = 0 \quad \text{or} \quad \overline{\mu} = \frac{\rho}{2} + 2\mu_1 \geq 2. \tag{4.4}$$

Since a plane quartic can have at most 6 nodes, in total we get $2 \leq \overline{\mu} \leq 6$ unless $\overline{D}$ is smooth.

Proposition 4.5 — *Let C be a nodal curve of arithmetic genus 3 with a fixed-point free involution τ, or equivalently, an étale map $\pi\colon C \to D$ to a nodal curve of arithmetic genus 2. Then the image of the canonical map $C \overset{|K_C|}{\dashrightarrow} \mathbb{P}^2$ is contained in a conic.*

In particular, C is not a plane curve.

Proof. By assumption there is a torsion divisor L on D such that $2L = 0$ defining the double cover and the projection formula gives a splitting

$$H^0(K_{\overline{D}}) = H^0(K_D) \oplus H^0(K_D + L).$$

Writing $H^0(K_D) = \langle x, y\rangle$ and $H^0(K_D + L) = \langle z\rangle$, we see that $z^2 \in H^0(2(K_D + L)) = H^0(2K_D) = \langle x^2, xy, y^2\rangle$, so there is a quadratic relation between the section defining the canonical map.

To conclude that C is not a plane curve note that a plane curve of arithmetic genus 3 is a plane quartic and hence canonically embedded by the adjunction formula. □

Corollary 4.6 — *In the above situation, the plane quartic $\overline{D}$ has at least three nodes.*

Proof. Let $\overline{D}$ be a nodal quartic with $\overline{\mu} \leq 2$ nodes. By (4.4) the case $\overline{\mu} = 1$ need not be considered.

We have to show, that an involution τ on $\overline{D}^\nu$ satisfying the Gorenstein glueing condition and yielding $\chi(D) = -1$ cannot exist. By Proposition 4.5 it is enough to show that such an involution would descend to a fixed-point-free involution on $\overline{D}$ itself.

This is clear if $\overline{D}$ is smooth as in this case $\overline{D} = \overline{D}^\nu$.

So we are left with $\overline{\mu} = 2$, in which case $\overline{D}^\nu$ is an elliptic curve with four marked points P_1, P_2, Q_1, Q_2 mapping to the nodes P and Q of $\overline{D}$. Assuming a suitable involution exists, by (4.4) the involution τ cannot have fixed points on $\overline{D}^\nu$ and all points P_i, Q_i map to a unique degenerate cusp.

As explained in Remark 4.3 this means that, up to renaming we have $\tau(P_i) = Q_i$. This is exactly the condition for τ to descend to a fixed-point-free involution on $\overline{D}$. By Proposition 4.5 this is impossible for a plane curve.

□

We now show explicitly that examples with three nodes exist.

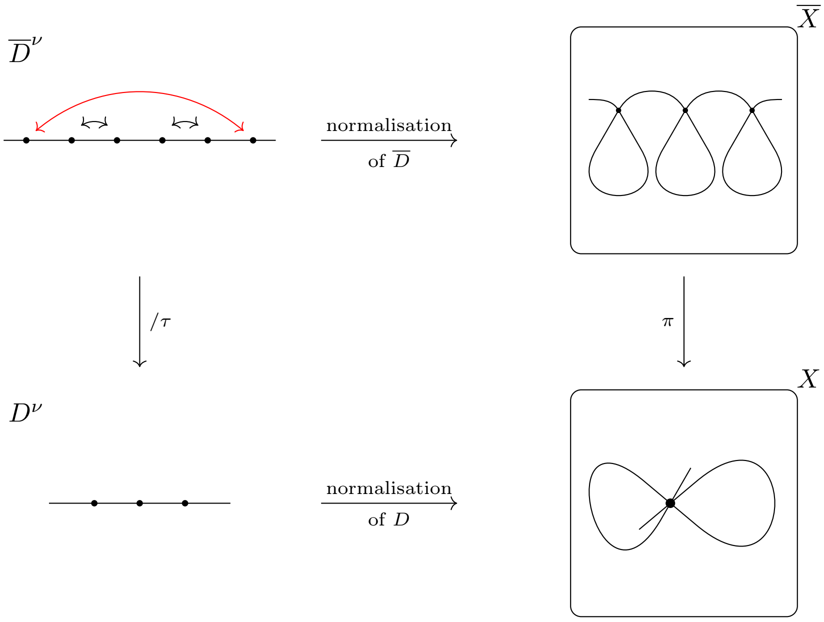


Figure 9: Case (P), $\overline{D}$ has three nodes

Example 4.7 ($\overline{D}$ irreducible with three nodes) — Assume $\overline{D}$ is an irreducible plane quartic with three nodes, which we may assume to be at $P_1 = (1:0:0), P_2 = (0:1:0)$ and $P_3 = (0:0:1)$. Its normalisation is $\overline{D}^{\nu} \cong \mathbb{P}^1$ with six marked points mapping to the nodes of $\overline{D}$. Assume that there is an involution τ restricting to a fixed-point-free involution in the marked points. Then one can choose coordinates $(u:v)$ such that $\tau(u:v) = (av:u)$ and the six points are

$$\begin{aligned} &(0:1), && \tau(0:1) = (1:0),\\ &(1:1), && \tau(1:1) = (a:1),\\ &(b:1), && \tau(b:1) = (a:b), \end{aligned}$$

for some $a, b \in \mathbb{C} \setminus \{0,1\}$ with $a \neq b$. If we denote the homogeneous coordinates of the projective plane with $(x:y:z)$ then each of these vanishes at four of the six points exactly once, thus determining which point maps to which node.

In order for the triple $(\mathbb{P}^1, \overline{D}, \tau)$ to satisfy the χ-condition, we infer from (4.4) that there is a uniqe degenerate cusp, that is, all six points are in the same class with respect to the equivalence relation explained in 4.3. Thus up to permuting the coordinates the map $\overline{\nu}\colon \overline{D}^{\nu} \to \mathbb{P}^2$ should be given by

$$\begin{aligned} x &= uv(u-v)(u-bv),\\ y &= u(u-v)(u-av)(bu-av),\\ z &= v(u-av)(u-bv)(bu-av), \end{aligned}$$

such that $\overline{\nu}^{-1}(P_1) = \{(a:1),(a:b)\}$, $\overline{\nu}^{-1}(P_2) = \{(1:0),(b:1)\}$, and $\overline{\nu}^{-1}(P_3) = \{(0:1),(1:1)\}$. Therefore, a curve admitting such an involution on the normalisation exists and the equation of the image can be computed using Macaulay2 to be

$$\begin{aligned} f_{a,b} = (-ab^3 + b^4 + a^2b - ab^2)x^2y^2 + {} & (a^2b^3 - a^3b - ab^3 - a^3 + 3a^2b - ab^2)x^2yz \\ & + (ab^2 - 2b^3 - a^2 + ab + b^2)xy^2z \\ & + (a^4 - a^3b - a^3 + a^2b)x^2z^2 \\ & + (2a^2b - ab^2 - a^2 - ab + b^2)xyz^2 \\ & + (b^2 - b)y^2z^2. \end{aligned}$$

By construction, the triple $(\mathbb{P}^2, \overline{D}, \tau)$ defines a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$, depending on the parameters a, b.

To sum up what we have done so far we state:

Proposition 4.8 — *Let X be a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(X) = 2$. If the normalisation $\overline{X} = \mathbb{P}^2$ and $\overline{D} \subset \overline{X}$ is irreducible, then X arises as in Example 4.7.*

Proof. Combining Corollary 4.6 and Lemma 4.2 we see that neccesarily $\overline{D}$ is a plane quartic with three non-colinear nodes.

As argued in Example 4.7 there is, up to the choice of coordinates only one way to pick the involution τ satisfying the χ-condition, in other words, X arises as in Example 4.7 as claimed. □

4.2.1. Case (P) with $\overline{D}$ reducible

From now on let $\overline{D}$ be a reducible quartic. The possiblities are given in Lemma 4.2. We treat the cases separatedly, starting with the most degenerate ones.

$\overline{D} =$ **four general lines** This case has been classified in [FPR15a, Sect. 4.2] and we follow their notation. Let $\overline{D} = L_1 + L_2 + L_3 + L_4$ be the union of four general lines. We denote $P_{(ij)}$ the intersection point of L_i and L_j. The normalisation of $\overline{D}$ is $\overline{D}^\nu = \sqcup L_i$ and we denote by P_{ij} the point of $L_i \in \overline{D}^\nu$ that maps to $P_{(ij)}$.

Since every component of $\overline{D}^\nu$ contains three such points, τ cannot preserve any of the L_i, so we may assume that it maps L_1 to L_2 and L_3 to L_4. Then τ is uniquely determined by two bijections

$$\varphi_{12}\colon \{P_{12}, P_{13}, P_{14}\} \to \{P_{21}, P_{23}, P_{24}\}.$$

$$\varphi_{34}\colon \{P_{31}, P_{32}, P_{34}\} \to \{P_{41}, P_{42}, P_{43}\}.$$

By loc. cit. X is isomorphic to one (and only one) of the surfaces $X_{2,1}, X_{2,2}$ and $X_{2,3}$ corresponding to the involutions listed in Table 7.

$\overline{D} =$ **a conic and two lines** The gluing involution τ has to preserve the conic and exchange the two lines, because they contain a different number of preimages of nodes. Thus we have five nodes in total, two fixed points of τ on the conic and thus by (4.4) exactly two degenerate cusps.

We will now determine all possible involutions τ, using the notation from Figure 12.

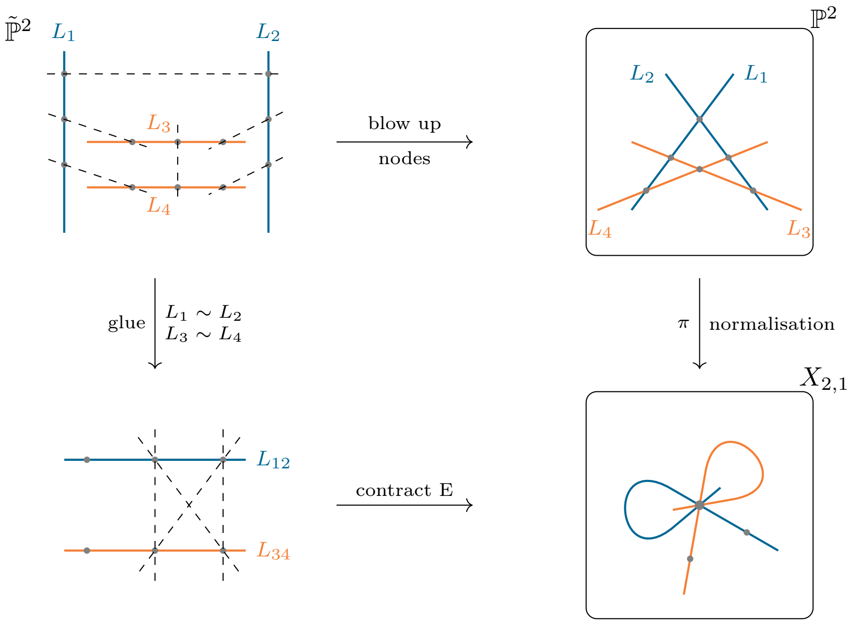


Figure 10: Construction of $X_{2,1}$ and $X_{2,2}$

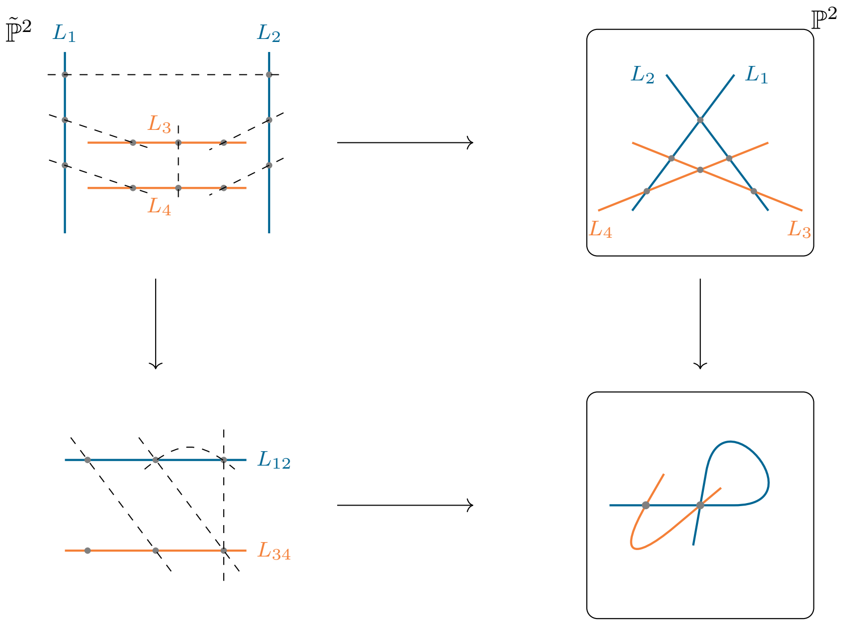


Figure 11: Construction of $X_{2,3}$

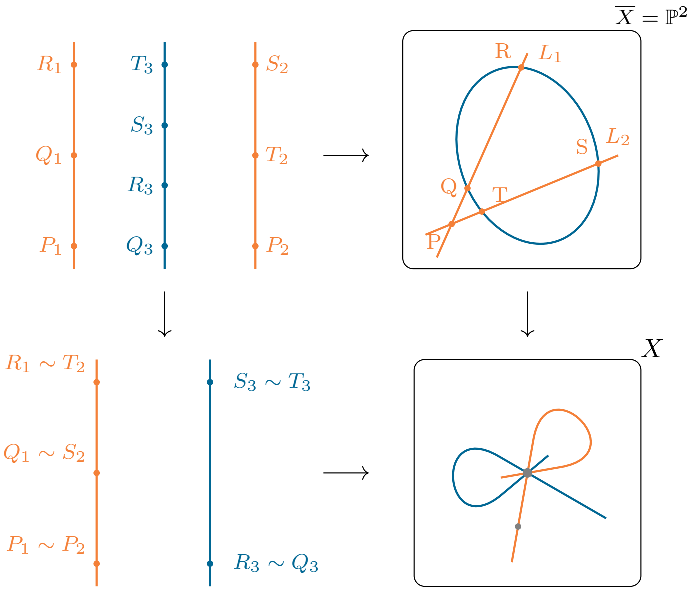


Figure 12: conic and two lines, Case A'

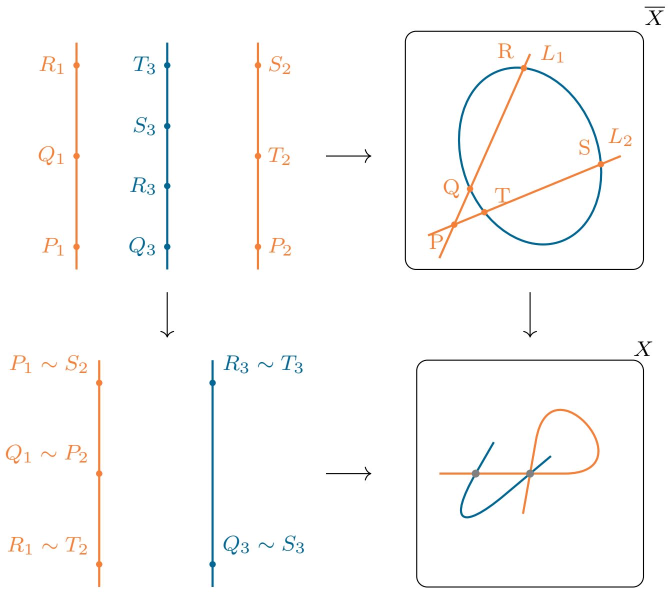


Figure 13: conic and two lines, Case B

Table 7: Surfaces from four lines in the plane from [FPR15b]

Surface	φ_{12} and φ_{34}	Degenerate cusps
$X_{2,1}$	$\varphi_{12} = \begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{21} & P_{24} & P_{23} \end{pmatrix}$	$\{P_{(12)}\}$, $\{P_{(34)}\}$,
	$\varphi_{34} = \begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{41} & P_{42} & P_{43} \end{pmatrix}$	$\{P_{(13)}, P_{(14)}, P_{(23)}, P_{(24)}\}$
$X_{2,2}$	$\varphi_{12} = \begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{21} & P_{23} & P_{24} \end{pmatrix}$	$\{P_{(12)}\}$, $\{P_{(34)}\}$,
	$\varphi_{34} = \begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{42} & P_{41} & P_{43} \end{pmatrix}$	$\{P_{(13)}, P_{(14)}, P_{(23)}, P_{(24)}\}$
$X_{2,3}$	$\varphi_{12} = \begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{23} & P_{24} & P_{21} \end{pmatrix}$	$\{P_{(12)}, P_{(23)}, P_{(14)}\}$,
	$\varphi_{34} = \begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{42} & P_{41} & P_{43} \end{pmatrix}$	$\{P_{(13)}, P_{(24)}\}$, $\{P_{(34)}\}$

Case A' **and Case** A''**:** Assume that $\tau(P_1) = P_2$, that is, the preimage of one degenerate cusp consists solely of the intersection point P of the two lines. Then all other preimages of nodes have to be equivalent under the equivalence relation of Remark 4.3 and there are up to renaming two possibilties τ' and τ'': either $\tau|_C$ preserves the intersection $L_i \cap C$, that is,

$$\tau'(Q_3) = R_3 \text{ and } \tau'(S_3) = T_3,$$

or it does not, that is,

$$\tau''(Q_3) = S_3 \text{ and } \tau''(R_3) = T_3.$$

In order to ensure the correct number of degenerate cusps we need to have

$$\tau' \mathrel{\hat{=}} \begin{pmatrix} P_1 & Q_1 & R_1 \\ P_2 & S_2 & T_2 \end{pmatrix} \text{ and } \tau'' \mathrel{\hat{=}} \begin{pmatrix} P_1 & Q_1 & R_1 \\ P_2 & T_2 & S_2 \end{pmatrix}.$$

These constructions depends on one parameter, namely the choice of the conic. If we degenerate the conic to a pair of lines, we arrive at $X_{2,1}$ or $X_{2,2}$ from Table 7.

Case B**:** If $\tau(P_1) \neq P_2$, then we can choose the involution on the lines to be given as

$$\begin{pmatrix} P_1 & Q_1 & R_1 \\ S_2 & P_2 & T_2 \end{pmatrix}.$$

and $\tau|_C(Q_3) = S_3$ and thus $\tau|_C(R_3) = T_3$ such that the preimages of the two degenerate cusps are $\{P, Q, S\}$ and $\{R, T\}$.

These constructions depend on one parameter, namely the choice of the conic. If we degenerate the conic to a pair of lines, we arrive at $X_{2,3}$ from Table 7.

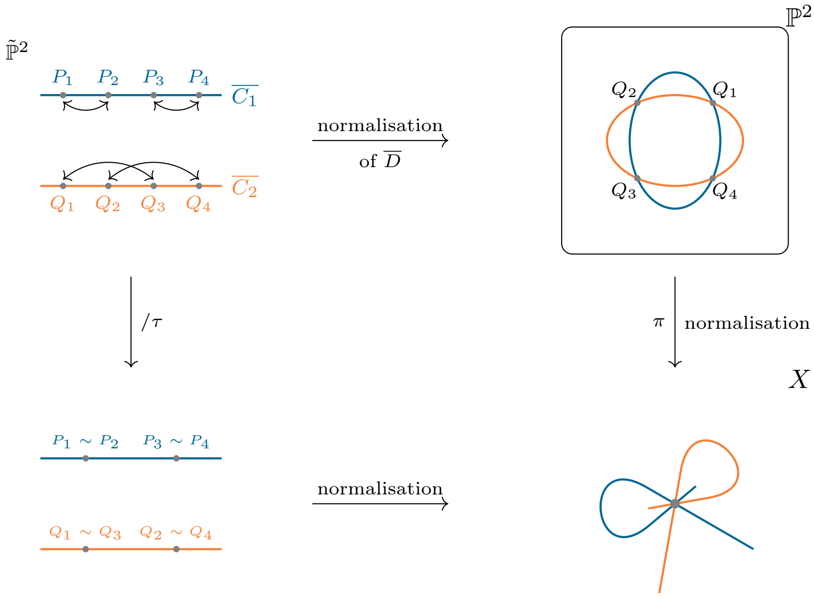


Figure 14: Case A: $\rho = 4$ and $\mu_1 = 1$

$\overline{D} =$ **two irreducible conics** By (4.4) there are two possibilities:

Case A: $\rho = 4$ **and** $\mu_1 = 1$ In this case the involution preserves the two components of $\overline{D}^\nu$ and, in order for there to be only one degenerate cusp, one can name the points such that the involution is as Figure 14.

It is an elementary fact, that given a projective line with four marked points, there is always an involution exchanging two pairs of points (compare Example 4.7), so the desired involutions exist on any two smooth conics in the pencil.

The construction depends on the choice of the two conics in a pencil, that is, two parameters. If we let one of the conics degenerate to a pair of lines then we can arrive the possibilites considerd in Case A' and Case A'' above. Making both conics reducible gives the surfaces $X_{2,1}$ and $X_{2,2}$ from Table 7.

Case C: $\rho = 0$ **and** $\mu_1 = 2$ Since an involution on $\mathbb{P}^1$ has fixed points, τ exchanges the components $\overline{D} = C' + C''$, that is, τ is induced by an abstract isomorphism $\varphi\colon C' \to C''$.

Let us denote the four intersection points of the two conics C', C'' with $Q_1, \ldots, Q_4$. We add the primes if we consider the points on the individual conics. By (4.4) we have a two degenerate cusps, say R_1 and R_2. Up to reindexing there are again two cases:

$\pi^{-1}(R_1) = \{Q_1\}$, $\pi^{-1}(R_2) = \{Q_2, Q_3, Q_4\}$**:** The abstract isomorphism φ

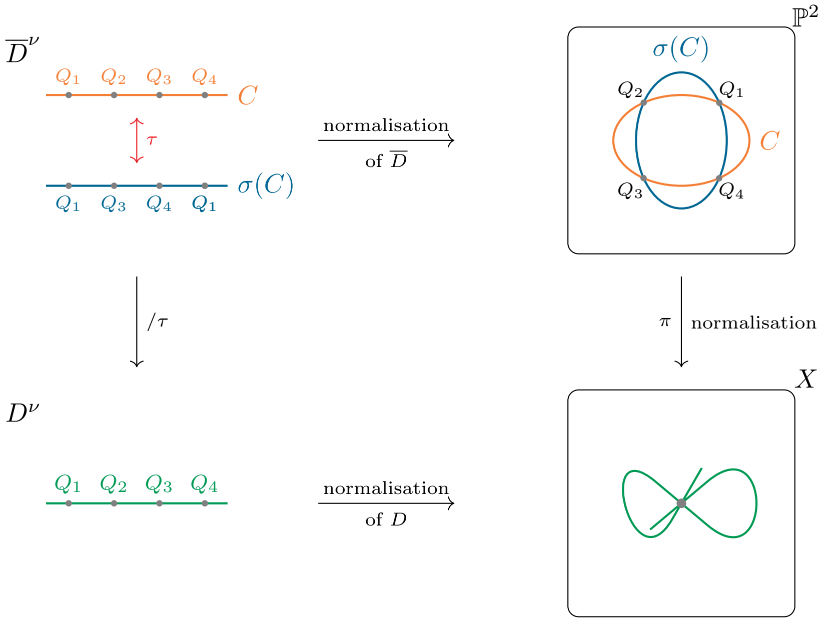


Figure 15: Case C: $\overline{D}$ = two irreducible conics, $\rho = 0, \mu_1 = 2$

has the property, up to reindexing again,

$$\varphi(Q_1') = Q_1'', \varphi(Q_2') = Q_3'', \varphi(Q_3') = Q_4'', \varphi(Q_4') = Q_2''.$$

Now consider the unique automorphism σ of $\mathbb{P}^2$ that acts on the Q_i, considered as point in the plane, in the same way as φ and let $\sigma C'$ be the image of C' under σ. The composition $\sigma \circ \varphi^{-1} \colon C'' \to \sigma C'$ is an abstract isomorphism of two plane conics fixing four points in the plane. By [FFP16, Es. 4.24] it is actually induced by the identity on $\mathbb{P}^2$, thus $C'' = \sigma C'$ and $\varphi = \sigma|_C'$.

Since C'' is determined by C', this construction depends on the one parameter. If we let $C' = L_1 + L_3$ the union of two lines, then the above construction still provides us with a suitable involution and it is straightforward to check, that it gives the case $X_{2,3}$ from Table 7.

$\pi^{-1}(R_1) = \{Q_1, Q_2\}$, $\pi^{-1}(R_2) = \{Q_3, Q_4\}$: Assume there is such an involution on $\overline{D}^\nu$. Then the involution descends to $\overline{D}$ itself violating Proposition 4.5.

Put differently, the argument used in Case A does not work because the morphism defined on the points will fix the given conic, see [FFP16, Es. 4.25].

Thus this case does not occur.

$\overline{D}$ = a smooth or nodal cubic and line The involution has to preserve the line, because either the number of marked points on the two components of $\overline{D}^\nu$ is different or they are not isomorphic. But on a line with three marked points τ cannot induce a fixed-point-free involution on the marked points in violation of the Gorenstein-condition.

Therefore this case cannot exist.

We have enumerated all possible cases for $\overline{D}$ and thus concluded the classification.

Proposition 4.9 — *Let X be a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(X) = 2$. If the normalisation $\overline{X} = \mathbb{P}^2$ and $\overline{D} \subset \overline{X}$ is reducible, then X arises as in Cases A, B, C in Section 4.2.1 or as a degeneration thereof.*

Remark 4.10 — By dimension reasons the surfaces constructed in Example 4.7 cannot degenerate to the general surface in Case A. It remains to work out explicitly their relation to the other reducible cases.

4.3. Case (dP) and (E_-)

The following is a reformulation of the results in [FPR15a]. We recall the enumeration of the two cases (dP) and (E_-)

(dP) $\overline{X}$ is a (possibly singular) Del Pezzo surface of degree 1, namely $\overline{X}$ has at most canonical singularities, $-K_{\overline{X}}$ is ample and $K_{\overline{X}}^2 = 1$. The curve $\overline{D}$ belong to the system $|-2K_{\overline{X}}|$, hence $K_{\overline{X}} + \overline{D} = -K_{\overline{X}}$ and $p_a(\overline{D}) = 2$.

(E_-) Let E be an elliptic curve and let $a : \tilde{X} \to E$ be a geometrically ruled surface that contains an irreducible section C_0 with $C_0^2 = -1$. Namely, $\tilde{X} = \mathbb{P}(\mathcal{O}_E + \mathcal{O}_E(-x))$, where $x \in E$ is a point and C_0 is the only one curve on the system $|\mathcal{O}_{\overline{X}}(1)|$. Set $F = a^{-1}(x)$: the normal surface $\overline{X}$ is obtained from $\tilde{X}$ by contracting C_0 to an elliptic Gorenstein singularity of degree 1 and $\overline{D}$ is the image of a curve $\overline{D}_0 \in |2(C_0 + F)|$ disjoint from C_0, so $p_a(\overline{D}) = 2$. The line bundle $K_{\overline{X}} + \overline{D}$ pulls back to $C_0 + F$ on $\tilde{X}$.

Lemma 4.11 — *Let $(\overline{X}, \overline{D})$ be a log-canonical pair such that $K_{\overline{X}} + \overline{D}$ is Cartier, $(K_{\overline{X}} + \overline{D})^2 = 1$, and the minimal resolution of $\overline{X}$ is either a del Pezzo surface of degree 1 or of type E_-. Then $-K_{\overline{X}}$ is an ample Cartier divisor of square 1 and $\overline{D} \in |-2K_{\overline{X}}|$.*

Moreover we have

$$R(\overline{X}, -K_{\overline{X}}) \cong \mathbb{C}[x_1, x_2, y, z]/(f_6)$$

with variables of degrees $(1, 1, 2, 3)$ and

$$f_6 = z^2 + a_0 y^3 + a_2 y^2 + a_4 y + a_6, \tag{4.12}$$

where $a_i = a_i(x_1, x_2)$ is of degree i.

If $\overline{D}$ is general in $|-2K_{\overline{X}}|$ then we can choose the coordinates such that $\overline{D} = \{y = 0\}$; the restriction of the anti-canonical ring to $\overline{D}$ gives a surjection

$$R(\overline{X}, -K_{\overline{X}}) \to \mathbb{C}[x_1, x_2, y, z]/(f_6, y) = \mathbb{C}[x_1, x_2, z]/(z^2 + a_6) = R(\overline{D}, K_{\overline{D}}).$$

Proof. Assume that $\overline{X}$ is a del Pezzo of degree 1. Then according to the description in the List 4.1, $-K_X$ is ample divisor and $K_{\overline{X}} + \overline{D} = -K$. We can easily compute the canonical ring using the method as in Section 1.2. Indeed, for any $m \geq 0$ and for all $i > 0$

$$H^i(X, -mK_X) = H^i(X, K_X + (-m-1)K_X) = 0$$

Riemann- Roch Theorem gives us

$$h^0(-mK_X) = \frac{1}{2}(-mK_X)(-mK_X - K_X) + 1 = \frac{m(m+1)}{2} + 1$$

Thus $h^0(-K_X) = 2, h^0(-2K_X) = 4$ and $h^0(-3K_X) = 7$. Let x_1, x_2 be generators of $H^0(-K_X)$, let y be element in $H^0(-2K_X)$ which is not in subspace generated by $S^2\langle x_1, x_2\rangle$, and let z be an element in $H^0(-3K_X)$ which is not in the subspace generated by $S^3\langle x_1, x_2\rangle \oplus \langle x_1 y, x_2 y\rangle$. By comparing the dimension of $H^0(-mK_X)$ and subspace generated by x_1, x_2, y, z we obtain a the relation $z^2 + a_0 y^3 + a_2 y^2 + a_4 y + a_6$ in degree 6. By using the similar argument as in Proposition 1.3, we conclude that there are no other relation for any $m > 6$. Thus the anti-canonical ring of a del Pezzo surface $\overline{X}$ is

$$R(\overline{X}, -K_X) \cong \mathbb{C}[x_1, x_2, y, z]/(f_6)$$

with variables of degrees $(1, 1, 2, 3)$ and $f_6 = z^2 + a_0 y^3 + a_2 y^2 + a_4 y + a_6$. Now we assume that $\overline{X}$ is of type E_-. In [FPR15a] the invariants of X were computed and they have the same invariants as in Case of del Pezzo surfaces. Thus we have the same canonical ring for Case (E_-) $\square$

Note that by the χ-condition, given a pair $(\overline{X}, \overline{D})$ as above, an involution τ on $\overline{D}^\nu$ defines a Gorenstein stable surface with $\chi(X) = 2$ if and only if it satisfies the Gorenstein glueing condition and the resulting curve D has arithmetic genus 1.

For simplicity, we restrict to the case where $\overline{D}$ is smooth, so that the Gorenstein glueing condition is automatically satisfied, and we are looking for curves $\overline{D}$ of genus 2 which admit an elliptic involution.

Lemma 4.13 — *Let $\overline{D}$ be a smooth curve of genus 2 admitting an elliptic involution τ, that is, $\overline{D}/\tau$ is an elliptic curve.*

Then decomposing pluricanonical forms into τ^*-eigenspaces allows to choose generators of the canonical ring such that

$$R(\overline{D}, K_{\overline{D}}) = \mathbb{C}[x_1, x_2, z]/(z^2 + a_6),$$

where x_1, x_2, z have degrees $1, 1, 3$ respectively,

$$a_6 = -(x_1^2 - \lambda_1 x_2^2)(x_1^2 - \lambda_2 x_2^2)(x_1^2 - \lambda_3 x_2^2), \quad \lambda_i \in \mathbb{C}^*, \tag{4.14}$$

and τ acts via $(x_1, x_2, z) \mapsto (-x_1, x_2, z)$.

Proof. If $\pi \colon \overline{D} \to \overline{D}/\tau = D$, then $\pi_* \mathcal{O}_{\overline{D}} = \mathcal{O}_D \oplus \mathcal{L}$ and $K_{\overline{D}} = \pi^*(K_D + L)$ and we get a decomposition on the cohomology by the projection formula. $\square$

Proposition 4.15 — *Let $(\overline{X}, \overline{D}, \tau)$ be the triple corresponding to a Gorenstein stable surface X with $K_X^2 = 1$ and $\chi(X) = 2$ and such that $\overline{X}$ is of type (dP) or (E_-).*

Then there exists a_6 as in (4.14) and f_6 as in (4.12) such that the inclusion map is induced by

$$\mathbb{C}[x_1, x_2, y, z]/(f_6) \to \mathbb{C}[x_1, x_2, z]/(z^2 + a_6)$$

and τ acts as in Lemma 4.13

Proof. Follows immediatly from the above lemma. □

We can now write down a family containing the surfaces discussed above as an open subset of:

$$\mathcal{W} = \{(a_0, a_2, a_4, a_6) \in \mathbb{C}[x_1, x_2] \mid a_6 \text{ as in } (4.14)\}$$

Proposition 4.16 — *The subset of $\overline{\mathfrak{M}}_{1,2}^{(Gor)}$ parametrising surfaces with normalisation (dP) or (E_-) is irreducible of dimension 10.*

Proof. Let $R = \mathbb{C}[x_1, x_2]$. Then this subset is dominated by an open subset of

$$\mathcal{W} = \{(a_0, a_2, a_4, a_6) \in R_0 \times R_2 \times R_4 \times R_6 \mid a_6 \text{ as in } (4.14)\},$$

which is of dimension $1 + 3 + 5 + 3 = 12$. The choices made in the above construction fix the coordinates up to multiplication with non-zero numbers. In addition, we fixed the coefficient in front of z^2 to be 1 as in (4.12) and the coefficient in front of x_1^6 in a_6 to be 1 as in (4.14). Thus we have a remaining action of $\mathbb{C}^{*2}$ by multiplication on x_2 and y, and the dimension of the stratum is $12 - 2 = 10$. □

4.4. Case (E_+)

Assume that X is a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$ and normalisation $\overline{X} = S^2E$ for an elliptic curve E. Using the notation from (4.1) we recall some facts from [FPR15b].

Consider

$$\begin{tikzcd} E \times E \arrow[r, "\sigma"] \arrow[dr, "\oplus"'] & S^2E \arrow[d, "\mathrm{alb}"] \\ & E \end{tikzcd}.$$

Then alb is a $\mathbb{P}^1$-bundle with section $C_0 = \{(0, p) \in S^2E \mid p \in E\}$ and the fibre over 0 is given by $F = \{(p, -p) \in S^2E \mid p \in E\}$. The canonical bundle is then $K_{S^2E} = -2C_0 + F$ and the conductor is a nodal curve of arithmetic genus 2

$$\overline{D} \in |3C_0 - F| = |C_0 - K_{S^2E}|.$$

By the χ-condition, the conductor in X has genus 0 and thus $\overline{D} \to D$ is the canonical map induced by the hyperelliptic involution (if $\overline{D}$ is smooth or at least irreducible).

For the sake of completeness we first recall an observation from [FPR17, Rem. 5.3].

Lemma 4.17 — *Let E be an elliptic curve and S^2E be its symmetric square.*

1. *There exist non-normal Gorenstein stable surfaces X with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 2$ and normalisation S^2E.*

2. *If X is such a surface then the bicanonical map $X \to \mathbb{P}^2$ is not a Galois cover.*

Proof. 1. A general element $\overline{D} \in |3C_0 - F|$ is smooth and, denoting by τ the hyperelliptic involution, the triple $(S^2E, \overline{D}, \tau)$ defines a surface with the required invariants by Kollár's glueing theorem as explained in Section 4.1.

2. As in [FPR17] Remark 5.3, a normalisation of a bi-double cover is again a bidouble cover, the canonical divisor of $\overline{X}$ is pullback of some $\mathcal{O}_{\mathbb{P}^2}(d)$ and thus either ample, anti ample or trivial. Thus no bi-double cover can have a normalisation of type E_+. □

Because of the second statement, a concrete (algebraic) description of an example could not be found in [FPR17]. We will now give such a description yielding in fact a complete family parametrising an open subset of the stratum of surfaces in $\overline{\mathfrak{M}}_{1,2}$ with normalisation the symmetric square of an elliptic curve.

We would like to compute the canonical ring of X which is based on the following result of Kollár:

Proposition 4.18 — *[Kol13][Prop. 5.8] Let X be a Gorenstein stable surface. Define the different $\Delta = Diff_{\overline{D}^\nu}(0)$ by the equality $(K_{\overline{D}} + \overline{D})|_{\overline{D}} = K_{\overline{D}} + \Delta$.*

Then a section $s \in H^0(\overline{X}, m(K_{\overline{X}} + \overline{D}))$ descends to a section in $H^0(X, mK_X)$ if and only if the image of s in $H^0(\overline{D}^\nu, m(K_{\overline{X}} + \overline{D}))$ under the Residue map is τ-invariant if m is even respectively τ- anti invariant if m is odd.

To compute the canonical ring of X, we need to compute the ring of sections

$$R(S^2E, K_{S^2E} + \overline{D}) = R(S^2E, C_0),$$

the residue map to $R(\overline{D}, K_{\overline{D}}) = R(\overline{D}, C_0|_{\overline{D}})$ including the action of the hyperelliptic involution. The strategy is to pull back to $E \times E$ and then to take invariants under the involution exchanging the factor. To simplify notation we add number to the factors $E \times E = E_1 \times E_2$.

We consider the geometric situation

$$\begin{tikzcd}
E_1 \times E_2 \arrow[r, hookleftarrow, "\tilde{\iota}"] \arrow[d, "\sigma"] & \tilde{D} \arrow[d] \\
S^2E \arrow[r, hookleftarrow, "\bar{\iota}"] \arrow[d, "\pi"] & \overline{D} \arrow[d, "\pi", "/\tau"'] \\
X \arrow[r, hookleftarrow, "\iota"] & D
\end{tikzcd} \tag{4.19}$$

For linear series or spaces of sections on $E_1 \times E_2$ we denote the invariant part under the involution interchanging the factors by a superscript $^+$.

Lemma 4.20 — *With the above notation we have*

$$\sigma^* C_0 = E_1 \times \{0\} + \{0\} \times E_2,$$
$$\sigma^* F = \Delta^- = \{(p, -p) \in E \times E \mid p \in E\},$$
$$\sigma^* \overline{D} = \tilde{D} \in \sigma^*|3C_0 - F| = |3\sigma^* C_0 - \Delta^-|^+.$$

We thus have

$$H^0(S^2E, mC_0) = H^0(E \times E, m\sigma^* C_0)^+ \cong (H^0(E_1, m \cdot 0) \otimes H^0(E_2, m \cdot 0))^+.$$

Therefore if $v_1, \ldots, v_m$ is a basis of $H^0(E, m{\cdot}0)$ then a basis of $(H^0(E_1, m{\cdot}0) \otimes H^0(E_2, m{\cdot}0))^+$ is given by

$$(v_i \otimes v_i)_{i=1,\ldots,m}, (v_i \otimes v_j + v_j \otimes v_i)_{1 \le i < j \le m}.$$

These are $m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2} = h^0(S^2E, mC_0) = \chi(S^2E, mC_0)$ elements as predicted by Riemann-Roch.

We now choose for the elliptic curve a Weierstrass type equation

$$f_i = y_i^2 - (x_i^3 + a x_i z_i^4 + b z_i^6)$$

such that $R(E_i, 0) \cong \mathbb{C}[z_i, x_i, y_i]/(f_i))$ with generators in degrees $(1, 2, 3)$.

Lemma 4.21 — *The low-degree parts of $R(S^2E, C_0)$, identified with the invariant subring of $R(E_1 \times E_2, \sigma^* C_0)$ are*

m	*Basis of* $H^0(S^2E, mC_0)$
1	$t_0 = z_1 z_2$
2	$t_0^2,\ t_1 = x_1 x_2, t_2 = z_1^2 x_2 + x_1 z_2^2$
3	$t_0^3 \oplus t_0\langle t_1, t_2\rangle,\ t_3 = y_1 y_2,\ t_4 = z_1 x_1 y_2 + y_1 z_2 x_2,\ t_5 = z_1^3 y_2 + y_1 z_2^3$
4	$t_0^4 \oplus t_0^2\langle t_1, t_2\rangle \oplus t_0\langle t_3, t_4, t_5\rangle \oplus S^2\langle t_1, t_2\rangle$, $t_6 = z_1 y_1 x_2^2 + x_1^2 z_2 y_2$

In fact, $t_0, \ldots, t_6$ generate the section ring $R(S^2E, mC_0)$.

Proof. If we follow for $m = 4$ the outlined procedure starting with $H^0(E, 4 \cdot 0) = \langle z^4, z^2x, zy, x^2\rangle$ we arrive at the listed elements, except that

$$z_1^4 x_2^2 + x_1^2 z_2^4 = t_2^2 - 2t_0^2 t_1.$$

To show that $t_0, \ldots, t_6$ generate the full invariant subring we argue as follows: Take a basis element given in the form

$$(z_1^{a_1} x_1^{b_1} y_1^{c_1})(z_2^{a_2} x_2^{b_2} y_2^{c_2}) + (z_1^{a_2} x_1^{b_2} y_1^{c_2})(z_2^{a_1} x_2^{b_1} y_2^{c_1}).$$

Since there are the relations f_i we may assume that $c_j \le 1$. Dividing by the generators t_0, t_1, t_3 we may also assume that $a_1 a_2 = b_1 b_2 = c_1 c_2 = 0$. The remaining possibilities are

$$x_1^a z_2^c + z_1^c x_2^a \equiv t_2^{d_2} t_5^{d_5} \mod (t_0, t_1) \qquad (c = 2d_2 + 3d_5)$$
$$x_1^a y_1 z_2^c + z_1^c x_2^a y_2 \equiv t_2^{d_2} t_5^{d_5} t_4 \mod (t_0, t_1) \qquad (c - 1 = 2d_2 + 3d_5)$$
$$x_1^a z_2^c y_2 + z_1^c y_1 x_2^a \equiv t_2^{d_2} t_5^{d_5} t_6 \mod (t_0, t_1) \qquad (c - 1 = 2d_2 + 3d_5)$$

so we have already found all generators in $t_0, \ldots, t_6$.

□

Taking the invariant part of a weighted Segre embedding we want to get the following diagram, where X is embedded in $\mathbb{P}(1,2,2,3,3)$ as complete intersection in degree $(6,6)$.

$$\begin{tikzcd}
\tilde{D} \arrow[r] \arrow[d] & E \times E \arrow[r, hook] \arrow[d, "\sigma"] & \mathbb{P}(1,2,3) \times \mathbb{P}(1,2,3) \arrow[d, dashed, "(t_0:\cdots:t_6)"] \\
\overline{D} \arrow[r] \arrow[d] & S^2E \arrow[r, hook, "R(C_0)"] \arrow[d, "\pi"] & \mathbb{P}(1,2,2,3,3,3,4) \arrow[d, dashed] \\
D \arrow[r] & X \arrow[r, hook, "R(K_X)"] & \mathbb{P}(1,2,2,3,3)
\end{tikzcd},$$

Lemma 4.22 — *Let s_F be a section defining F. Then the image of the composition*

$$H^0(S^2E, \overline{D}) \xhookrightarrow{\cdot s_F} H^0(S^2E, 3C_0) \hookrightarrow H^0(E \times E, \sigma^* 3C_0)$$

is spanned by the sections $t_4 = z_1x_1y_2 + y_1x_2z_2, t_5 = y_1z_2^3 + z_1^3y_2$.

Proof. Consider the exact sequence

$$0 \to H^0(E \times E, \sigma^*\overline{D}) \to H^0(E \times E, 3\sigma^*C_0) \to H^0(E \times E, \sigma^*3C_0|_{\Delta^-}).$$

Since in the Weierstrass model inversion on the elliptic curve corresponds to changing the sign of the y-coordinate one can see that the only invariant sections of σ^*3C_0 vanishing on Δ^- are the ones given above. □

We now fix the section $s_{\overline{D}}$ defining $\overline{D}$. By Lemma 4.22 there exist $\alpha, \beta \in \mathbb{C}$ such that the image of $s_{\overline{D}}$ in $H^0(S^2E, 3C_0)$ is

$$s_{\overline{D}} \cdot s_F = l = \alpha t_4 + \beta t_5.$$

Theorem 4.23 — *The canonical ring of X is considered as a subring of $R(E \times E, \sigma^*C_0)$, and it is generated by $s_0, \dots s_3, s_4 = l)$ with equations 4.24.*

$$\begin{aligned} Z_1^2 + b_1(X, Y_1, Y_2) &= 0 \\ Z_2^2 + XZ_1a_2(X, Y_1, Y_2) + b_2(X, Y_1, Y_2) &= 0 \end{aligned} \tag{4.24}$$

with

$$\begin{aligned}
b_1 &= -\left(b^2X^6 + abX^4Y_1 + bY_1^3 + a^2X^4Y_2 - 3bX^2Y_1Y_2 + aY_1^2Y_2 - 2aX^2Y_2^2 + Y_2^3\right) \\
a_2 &= -\left(2\beta^2X^2 + 2\alpha\beta Y_1 + 2\alpha^2Y_2\right) \\
b_2 &= -\left(2b\beta^2X^6 + \left(2b\alpha\beta + a\beta^2\right)X^4Y_1 + b\alpha^2X^2Y_1^2 + \beta^2Y_1^3 + \left(-2b\alpha^2 + 4a\alpha\beta\right)X^4Y_2\right. \\
&\quad \left. + (a\alpha^2 - 3\beta^2)X^2Y_1Y_2 + 2\alpha\beta Y_1^2Y_2 - 4\alpha\beta X^2Y_2^2 + \alpha^2Y_1Y_2^2\right)
\end{aligned}$$

Therefore X is an iterated double cover.

4.4.1. Proof of Theorem 4.23

By Kollár's result (Proposition 4.18), the canonical ring of X is the pullback ring in the diagram

$$\begin{tikzcd} R(E\times E,\sigma^*C_0)^+ = R(S^2E,C_0) \arrow[r, equal] & R(S^2E,K_{S^2E}+\overline{D}) \arrow[r, "\overline{\iota}^*"] & R(\overline{D},K_{\overline{D}}) \\ & R(X,K_X) \arrow[u, hook] \arrow[r, hook] & R(D,K_X|_D) \arrow[u, hook, "\pi^*"] \end{tikzcd}$$

In the following we identify $R(S^2E,K_{S^2E}+\overline{D}) = R(E\times E,\sigma^*C_0)^+$ via the pullback map σ^*.

Lemma 4.25 — *Let $m\geq 2$. Let $s_{\overline{D}}$ be the section defining $\overline{D}$ and s_F be the section defining F, so that $s_{\overline{D}}s_F = l$. Then the sequence*

$$0\to H^0(S^2E,mC_0-\overline{D}) \overset{s_{\overline{D}}}{\to} H^0(S^2E,mC_0)\to H^0(\overline{D},mK_{\overline{D}})\to 0$$

is exact. In particular, $h^0(S^2E,mC_0-\overline{D}) = \frac{m(m+1)}{2}-(2m-1) = \frac{m(m-3)}{2}+1$.

m	*generators of image of* $H^0(S^2E,mC_0-\overline{D})$ *in* $H^0(S^2E,mC_0)$
1	0
2	0
3	$l=\alpha t_4+\beta t_5$
4	$t_0 l$,
	$l_1=(b\alpha-a\beta)t_0^4+\beta t_0^2t_2-\beta t_1^2-\alpha t_1t_2-\alpha t_0t_3$,
	$l_2=b\beta t_0^4+a\alpha t_0^2t_2+b\alpha t_0^2t_1-\alpha t_2^2-\beta t_1t_2+\beta t_0t_3$

Proof. M2, reference. ☐

Lemma 4.26 — *1. The canonical ring of $\overline{D}$ is $R(\overline{D},K_{\overline{D}})=\mathbb{C}[A,B,C]/(h)$ where A,B have degree 1, C has degree 3 and $h=C^2+f(A,B)$ has degree 6.*

2. The image of $\pi^*\colon R(D,K_X|_D)\to R(\overline{D},K_{\overline{D}})$ is the subring $\mathbb{C}[A,B]$.

3. The image of the residue map $R(S^2E,C_0)\to R(\overline{D},K_{\overline{D}})$ is the subring generated by A,AB,B^2,B^3,C,CB after appropriate choice of coordinates.

Proof. Recall that the non-normal locus D is isomorphic to $\mathbb{P}^1$ and the map $\overline{D}\to D$ is the canonical map. It is well known that for a curve $\overline{D}$ of genus 2 , then $\omega_{\overline{D}}$ has degree $2g-2=2$. $\omega_{\overline{D}}$ has 2 sections and it is base-point-free, thus the canonical map induces a double cover $\overline{D}\to D$, where $D=\mathbb{P}^1$ branched over 6 points. Thus the canonical ring of $\overline{D}$ is of the form $R(K_{\overline{D}})=\mathbb{C}[A,B,C]/(C^2+f(A,B))$ with generators in degrees (1,1,3), where f is a homogeneous polynomial of degree 6. From this cover map, it is easy to see that $\pi^*\colon R(D,K_X|_D)\to R(\overline{D},K_{\overline{D}})$ corresponds to $\mathbb{C}[A,B]\to\mathbb{C}[A,B,C]/(C^2-f(A,B))$ and thus its image is the subring of $\mathbb{C}[A,B]$.

Moreover, consider the residue map $R(S^2E,K_{S^2E}+\overline{D})\to R(\overline{D},K_{\overline{D}})$. We can compute the image of this residue map as following. For $m=1$, consider the exact sequence

$$0 \to K_{S^2E} \to K_{S^2E} + \overline{D} = C_0 \to (K_{S^2E} + \overline{D})|_{\overline{D}} = K_{\overline{D}} \to 0$$

We get

$$0 \to H^0(K_{S^2E}) \to H^0(C_0) \to H^0(K_{\overline{D}}) \to H^1(K_{S^2E}) \to H^1(C_0) = 0$$

where $H^0(K_{S^2E}) = 0; h^0(C_0) = 1$ and $h^0(K_{\overline{D}}) = 2$. Thus the map $H^0(C_0) \to H^0(K_{\overline{D}})$ is injective, we can choose the first element A which is the image of $H^0(C_0)$ in $H^0(K_{\overline{D}})$.

For $m \geq 2$:

$$H^0(S^2E, m(K_{S^2E} + \overline{D})) \to H^0(\overline{D}, mK_{\overline{D}}) \to H^1(m(K_{S^2E} + \overline{D}) - \overline{D})$$

We have $H^1(m(K_{S^2E} + \overline{D}) - \overline{D}) = H^1((m-1)C_0 + K_{S^2E})$. Since $(m-1)C_0$ is ample if $m-1$ is positive, $H^1((m-1)C_0 + K_{S^2E}) = 0$ for $m \geq 2$ by Kodaira vanishing. Thus the map $H^0(S^2E, m(K_{S^2E} + \overline{D})) \to H^0(\overline{D}, mK_{\overline{D}})$ is surjective if and only if $m \geq 2$. Thus we get more images AB, B^2, B^3, C, CB. The elements A^2, A^2B, AB^2 are singled out by the relations $A^2 = A.A, A^2B = A.AB, AB^2 = A.B^2$. Thus the image of the residue map is generated by A, AB, B^2, B^3, C, CB. □

Lemma 4.27 — *There exists a choice of generators A, B of degree 1 and C of degree 3 such that the diagram*

$$\begin{tikzcd}
R(E \times E, \sigma^* C_0)^+ \arrow[r, "\bar{\iota}^*"] & R(\overline{D}, K_{\overline{D}}) \arrow[r, equal] & \mathbb{C}[A,B,C]/(C^2 - g(A,B)) \\
& R(D, K_X|_D) \arrow[u, hook] \arrow[r, equal] & \mathbb{C}[A,B] \arrow[u, hook]
\end{tikzcd} \tag{4.28}$$

commutes and $\bar{\iota}^* t_0 = A$.

Proof. The fact that the right hand side of the diagram is of the given form follows from the fact that $\overline{D}$ is a hyperelliptic curve (or just of genus 2) and $\overline{D} \to D$ is the quotient by the hyperelliptic involution.

Considering only the part of degree 1 in the rings we have

$$H^0(E \times E, \sigma^* C_0)^+ = \langle t_0 \rangle \xhookrightarrow{\bar{\iota}^*} \langle A, B \rangle$$

and we can arrange $\bar{\iota}^* t_0 = A$ by a linear coordinate change not affecting the form of the equation for $\overline{D}$. □

Lemma 4.29 — *In Diagram (4.28) consider the elements of degree 2 giving*

$$\langle t_0^2, t_1, t_2 \rangle = H^0(S^2E, 2C_0) \xrightarrow[\cong]{\bar{\iota}^*} H^0(\overline{D}, 2K_{\overline{D}}) = \langle A^2, AB, B^2 \rangle.$$

Let $s_0 = t_0$, $s_1 = \alpha t_2 + \beta t_1$ and $s_2 = (2b\alpha\beta - a\beta^2)t_0^2 + b\alpha^2 t_1 + (a\alpha^2 + \beta^2)t_2$. Then

$$AB = \bar{\iota}^*(s_1), \text{ and } B^2 = \bar{\iota}^*(s_2).$$

Proof. Since the map is an isomorphism and by the choice of $\overline{\iota}^* t_0 = A$ from Lemma 4.27 we need to show that there is an essentially unique way to complete s_0^2 to a basis s_0^2, s_1, s_2 of $\langle t_0^2, t_1, t_2\rangle$ such that $s_0^2 s_2 - s_1^2 = 0$ after restriction to $\overline{D}$.

We compute this inside $H^0(S^2E, 4C_0) = H^0(E \times E, 4\sigma^* C_0)^+$. By Lemma 4.25, we look at the sections of degree 4 and see that there is unique way to kill the term $t_0 t_3$ on the last two generetors by taking $\beta l_1 + \alpha l_2$, and write it on the form $s_0^2 s_2 - s_1^2$ modulo the equations of $\overline{D}$. We get the following equivalent class :

$$t_0^2((2b\alpha\beta - a\beta^2)t_0^2 + b\alpha^2 t_1 + (a\alpha^2 + \beta^2)t_2) = (\alpha t_2 + \beta t_1)^2$$

Then $s_1 = \alpha t_2 + \beta t_1$ and $s_2 = (2b\alpha\beta - a\beta^2)t_0^2 + b\alpha^2 t_1 + (a\alpha^2 + \beta^2)t_2$ do the job. $\square$

Lemma 4.30 — *In Diagram* (4.28) *consider the elements of degree* 3 *giving*

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0(S^2E, 3C_0 - \overline{D}) & \longrightarrow & H^0(S^2E, 3C_0) & \xrightarrow{\overline{\iota}^*} & H^0(\overline{D}, 3K_{\overline{D}}) & \longrightarrow 0 \\
 & \| & & \| & & \| & \\
0 \longrightarrow & \langle t_4 + 14t_5\rangle & \hookrightarrow & \langle s_0^3, s_0 s_1, s_0 s_2, t_3, t_4, t_5\rangle & \twoheadrightarrow & \langle A^3, A^2B, AB^2, B^3, C\rangle & \longrightarrow 0 \\
 & \| & & \uparrow & & \uparrow & \\
0 \longrightarrow & \langle t_4 + 14t_5\rangle & \hookrightarrow & H^0(X, 3K_X) & \xrightarrow{\iota^*} & \langle A^3, A^2B, AB^2, B^3\rangle & \longrightarrow 0
\end{array}$$

With the choices of s_0, s_1, s_2 *from Lemma 4.27 and Lemma 4.29 the element*

$$s_3 = -9t_0^3 + 44t_0 t_1 + 42 t_0 t_2 + 20 t_3$$

satisfies $\overline{\iota}^* s_3 = B^3$ *and it is unique with this property modulo* $s_{\overline{D}}$.

Proof. In Diagram (4.28) consider the elements of degree 3, this gives us

$$\langle s_0^3, s_0 s_1, s_0 s_2, t_3, t_4, t_5\rangle = H^0(S^2E, 3C_0) \xrightarrow{\overline{\iota}^*} H^0(\overline{D}, 3K_{\overline{D}}) = \langle A^3, A^2B, AB^2, B^3, C\rangle.$$

By previous lemmas we already identified s_0, s_1, s_2 and their images A, AB, B^2. Note that C has no relations with other generators and clearly $t_4 + 14t_5$ is in the image because it restricts to zero on $\overline{D}$. Thus there is only way to map $t_4 + 14t_5$ to C. We need only to identify the element $s_3 \in \langle t_0^3, t_0 t_1, t_0 t_2, t_3\rangle$ such that $s_0^3 s_3 - s_0^2 s_1 s_2 = 0$ modulo the equations of $\overline{D}$. We can also do this in degree 4 by relation $s_0 s_3 - s_1 s_2 = 0$ after restriction to $\overline{D}$. From the equations of s_1, s_2, l_1, l_2 it is easy to see that $s_1 s_2 + b\alpha^2 l_1 + (a\alpha^2 + \beta^2) l_2$ kills all terms of $t_1^2, t_1 t_2, t_2^2$ and equals to

$$t_0((b^2\alpha^3 + b\beta^3)t_0^3 + (ab\alpha^3 + 3b\alpha\beta^2 - a\beta^3)t_0 t_1 + (a^2\alpha^3 + 3b\alpha^2\beta)t_0 t_2 + (-b\alpha^3 + a\alpha^2\beta + \beta^3)t_3)$$

Then $s_3 = (b^2\alpha^3 + b\beta^3)t_0^3 + (ab\alpha^3 + 3b\alpha\beta^2 - a\beta^3)t_0 t_1 + (a^2\alpha^3 + 3b\alpha^2\beta)t_0 t_2 + (-b\alpha^3 + a\alpha^2\beta + \beta^3)t_3$ satisfied $s_0 s_3 - s_1 s_2 = 0$ modulo $\overline{D}$ and $\overline{\iota}^* s_3 = B^3$.

$\square$

Remark 4.31 — In Diagram 4.28 consider the elements of degree 4 giving

$$H^0(S^2E, 4C_0) \xrightarrow{\overline{\iota}^*} H^0(\overline{D}, 4K_{\overline{D}}) = \langle A^4, A^3B, A^2B^2, AB^3, B^4, AC, BC\rangle.$$

We need to identify BC with an element of $s_5 \in H^0(S^2E, 4C_0)$ so that $\overline{\iota}^*(s_5) = BC$. Since BC has no relation with other elements, there is only one way to identify $s_5 = t_6$.

Remark 4.32 — By the descriptions above, we have identified all the s_i. The result is a weighted projective space $\mathbb{P}(1,2,2,3,3,4)$. However, we recall that the structure of the canonical ring of X from Section 1.2, we would like to find two equations in bi-degree $(6,6)$ of X. This can be found by considering the projection away from s_5 as the following diagram

$$
\begin{tikzcd}
\tilde{D} \arrow[r] \arrow[d] & E \times E \arrow[r, hook] \arrow[d, "\sigma"] & \mathbb{P}(1,2,3) \times \mathbb{P}(1,2,3) \arrow[d, dashed, "(t_0:\cdots:t_6)"] \\
\overline{D} \arrow[r] \arrow[d] & S^2E \arrow[r, hook, "R(C_0)"] \arrow[d, "\pi"] & \mathbb{P}(1,2,2,3,3,3,4) \arrow[d, dashed, "(s_0:\cdots:s_5)"] \\
D \arrow[r] & X \arrow[r, hook, "R(K_X)"'] \arrow[dr] & \mathbb{P}(1,2,2,3,3,4) \arrow[d, dashed] \\
 & & \mathbb{P}(1,2,2,3,3)
\end{tikzcd}
.
$$

Thus the two equations in be-degree (6,6) can be computed using Macaulay2 [GS02].

A. Appendix

In this section, we use Macaulay2 to compute the canonical ring of the last case $R(S^2E)$. We start by setting up the polynomial rings with rational coefficients. The ambient space which corresponds to the projective space $\mathbb{P}(1,2,2,3,3,3,4)$ and the weighted Segre embedding $\mathbb{P}(1,2,3) \times \mathbb{P}(1,2,3) \to \mathbb{P}(1,2,2,3,3,3,4)$. We can also identify the generators of $R(C_0)$ in low degree, which help us to find generators of the canonicl ring of X.

```
P = QQ
Q = P[a,b,c, alpha, beta,  Degrees=>{1,1,1,1,1}]
Sambient = Q**P[t_0..t_6,  Degrees
=>{{1,1},{2,2},{2,2},{3,3},{3,3},{3,3}, {4,4}}];
```

$S(...)$ gives the weighted polynomial rings of the ambient spaces, which are related by projections. The canonical ring to be computed is the complete intersection of two homogeneous polynomials of degree 6 in SX.

```
SX = P[X,Y_1, Y_2,Z_1, Z_2, T, Degrees=>{1,2,2,3,3,4}];
```

SE1 and SE2 give us the ambient space of the elliptic curve $E_i, i = 1,2$, such that $R(E_i,0) \cong \mathbb{C}[z_i,x_i,y_i]/(f_i)$

```
SE1 = P[z_1, x_1, y_1, Degrees=>{1,2,3}]
SE2 = P[z_2, x_2, y_2, Degrees=>{1,2,3}]
SExE= SE1**SE2
-- equations for elliptic curves E
f1 = c*x_1^3+a*x_1*z_1^4+b*z_1^6;
f2 = c*x_2^3+a*x_2*z_2^4+b*z_2^6;
-- ideal of E\times E
i = ideal (c*y_1^2-f1, c*y_2^2-f2);
SExEi = SExE/i
```

The invariants under involution

```
tt =  matrix{{z_1*z_2,
      x_1*z_2^2+z_1^2*x_2,
      x_1*x_2,
      y_1*y_2,
      z_1*x_1*y_2+y_1*x_2*z_2,
      y_1*z_2^3+z_1^3*y_2,
      z_1*y_1*x_2^2+x_1^2*y_2*z_2}};
f = map (SExEi, Sambient, sub(tt,SExEi));
iS2E = ideal mingens ker f--this is the fiber.
```

Furthermore, we have the equation for $\overline{D}+F$. We denote F is the class of fiber of the Albanese map $a : S^2E \to E$, and $\overline{D}$ is the non-normal locus in S^2E, and l is section in $H^0(S^2E, 3C_0)$ defining $\overline{D}$

```
l = alpha*t_4+beta*t_5
D = saturate(ideal mingens iS2E+l,ideal(t_4,t_5))
F = saturate(ideal mingens(iS2E+l+ideal(t_4,t_5)))
assert( 5 == codim F)
assert (5 == codim D)
SbarD = Sambient/D
SS2E = Sambient/iS2E
barD = ideal mingens sub(D,SS2E)
```

We want to identify the s_i, the generators of $\mathbb{P}(1,2,2,3,3,4)$.

```
--degree 2, find the s1,s2
L = flatten entries mingens sub(D,SS2E)
L_0
L1= sub(L_1, c=>1)
L2=sub(L_2, c=>1)
LL=beta*L_1+alpha*L_2
s1=alpha*t_2+beta*t_1
s2=(sub(LL,c=>1)-sub(LL, {c=>1,t_0=>0}) )//t_0^2
s3 = s1*s2+b*alpha^2*L1+(a*alpha^2+beta^2)*L2
```

The following expresses the two relations of the canonical ring. This gives us precisely two homogeneous polynomials of degree 6, which provides us with the canonical ring of R. We write $R(K_X)$.

```
SE=Sambient/ker f;
ss =  matrix{{t_0,
alpha*t_2+beta*t_1,
a*alpha^2*t_2+2*b*alpha*beta*t_0^2-a*beta^2*t_0^2+b*alpha^2*
t_1+beta^2*t_2,
b^2*alpha^3*t_0^3+a*b*alpha^3*t_1*t_0+a^2*alpha^3*t_2*t_0+b*
beta^3*t_0^3+3*b*alpha*beta^2*t_1*t_0-a*beta^3*t_1*t_0+3*b*
alpha^2*beta*t_2*t_0-b*alpha^3*t_3+a*alpha^2*beta*t_3+beta^3*
t_3,
l}};
g=map(SE,SX,sub(ss,Sambient)); ker g.
```

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